

Stabilities of affine Legendrian submanifolds and their moduli spaces

Kotaro Kawai *

Abstract

We introduce the notion of affine Legendrian submanifolds in Sasakian manifolds and define a canonical volume called the ϕ -volume as odd dimensional analogues of affine Lagrangian (totally real or purely real) geometry. Then we derive the second variation formula of the ϕ -volume to obtain the stability result in some η -Einstein Sasakian manifolds. It also implies the convexity of the ϕ -volume functional on the space of affine Legendrian submanifolds.

Next, we introduce the notion of special affine Legendrian submanifolds in Sasaki-Einstein manifolds as a generalization of that of special Legendrian submanifolds. Then we show that the moduli space of compact connected special affine Legendrian submanifolds is a smooth Fréchet manifold.

1 Introduction

Affine Lagrangian (totally real or purely real) submanifolds are “maximally non-complex” submanifolds in almost complex manifolds defined by relaxing the Lagrangian condition (Definition 3.1). The affine Lagrangian condition is an open condition and hence there are many examples. Borrelli [2] defined a canonical volume of an affine Lagrangian submanifold called the J -volume. He obtained the stability result for the J -volume as in the Lagrangian case [4]. Lotay and Pacini [8] pointed out the importance of affine Lagrangian submanifolds in the study of geometric flows. Opozda [11] showed that the moduli space of (special) affine Lagrangian submanifolds was a smooth Fréchet manifold.

In this paper, we study the odd dimensional analogue. First, we introduce the notion of affine Legendrian submanifolds in Sasakian manifolds and define a canonical volume called the ϕ -volume as odd dimensional analogues of affine Lagrangian geometry. See Definitions 3.8 and 3.12. Then we compute the first variation of the ϕ -volume and characterize a critical point for the ϕ -volume by the vanishing of some vector field H_ϕ (Proposition 4.5), which is a generalization of the mean curvature vector (Remark 4.6). We call an affine Legendrian submanifold ϕ -**minimal** if $H_\phi = 0$. Then we compute the second variation of the ϕ -volume and obtain the following.

Theorem 1.1. *Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$ -dimensional Sasakian manifold and $\iota : L^n \hookrightarrow M$ be an affine Legendrian immersion of a compact oriented*

*The author is supported by Grant-in-Aid for JSPS fellows (26-7067).

n -dimensional manifold L . Let $\iota_t : L \hookrightarrow M$ be a one-parameter family of affine Legendrian immersions satisfying $\iota_0 = \iota$. Suppose that $\frac{\partial \iota_t}{\partial t}|_{t=0} = Z = \phi Y + f\xi$, where $Y \in \mathfrak{X}(L)$ is a vector field on L and $f \in C^\infty(L)$ is a smooth function. Then we have

$$\begin{aligned} \frac{d^2}{dt^2} \int_L \text{vol}_\phi[\iota_t] \Big|_{t=0} &= \int_L ((2n+2)\eta(Y)^2 - 2g(Y, Y) - \text{Ric}(Y, Y) \\ &\quad - g(\pi_L[Z, Y], H_\phi) + g(Y, H_\phi)^2 + \left(\frac{\text{div}(\rho_\phi[\iota]Y)}{\rho_\phi[\iota]} \right)^2) \text{vol}_\phi[\iota], \end{aligned}$$

where $\text{vol}_\phi[\iota]$ is the ϕ -volume form of ι given in Definition 3.12, Ric is the Ricci curvature of (M, g) , $\pi_L : \iota^*TM \rightarrow \iota_*TL$ is the canonical projection given in (3.3), $\rho_\phi[\iota]$ is the function on L given in Definition 3.12 and H_ϕ is the vector field on L given in Definition 4.3.

Remark 1.2. For Legendrian submanifolds, the ϕ -volume agrees with the standard Riemannian volume (Lemma 3.14). When ι is minimal Legendrian and all of ι_t 's are Legendrian, Theorem 1.1 agrees with [10, Theorem 1.1]. When ι is Legendrian-minimal Legendrian and all of ι_t 's are Legendrian, Theorem 1.1 agrees with [7, Theorem 1.1]. See Remark 5.3.

Following the Riemannian case, we call a ϕ -minimal affine Legendrian submanifold ϕ -**stable** if the second variation of the ϕ -volume is nonnegative.

Now, suppose that a $(2n+1)$ -dimensional Sasakian manifold $(M^{2n+1}, g, \eta, \xi, \phi)$ is a η -Einstein with the η -Ricci constant $A \in \mathbb{R}$. (See Definition 2.5.) Then we obtain the following.

Theorem 1.3. *Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$ -dimensional η -Einstein Sasakian manifold with the η -Ricci constant $A \leq -2$. Then any ϕ -minimal affine Legendrian submanifold in M is ϕ -stable.*

This is a generalization of [10, Theorem 1.2]. The author obtained further results by restricting variations of a minimal Legendrian submanifold to its Legendrian variations. In our case, since the affine Legendrian condition is an open condition, any small variation is affine Legendrian. Thus there is no analogue of these results.

Similarly, using the notion of convexity in the space of affine Legendrian submanifolds (Definition 3.17), we easily see the following.

Theorem 1.4. *Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$ -dimensional η -Einstein Sasakian manifold with the η -Ricci constant $A \leq -2$. Then the ϕ -volume functional on the space of affine Legendrian submanifolds is convex.*

For affine Legendrian submanifolds in a η -Einstein Sasakian manifold with the η -Ricci constant $A > -2$, we have the following.

Theorem 1.5. *Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$ -dimensional η -Einstein Sasakian manifold with the η -Ricci constant $A > -2$. Then there are no ϕ -minimal affine Legendrian submanifolds which are ϕ -stable.*

Next, we define a special affine Legendrian submanifold in a Sasaki-Einstein manifold with a Calabi-Yau structure on its cone by requiring that its cone is

special affine Lagrangian (Definition 6.13). This notion is a generalization of that of special Legendrian submanifolds. By a slight generalization of the general deformation theory of Moriyama [9, Proposition 2.2], we study the moduli space of special affine Legendrian submanifolds and obtain the following.

Theorem 1.6. *Let M be a Sasakian manifold with a Calabi-Yau structure on its cone and L be a compact connected manifold admitting a special affine Legendrian embedding $L \hookrightarrow M$. Then the moduli space of special affine Legendrian embeddings of L is an infinite dimensional smooth Fréchet manifold modeled on the Fréchet vector space $\{(g, \alpha) \in C^\infty(L) \oplus \Omega^1(L); (n+1)g + d^*\alpha = 0\} \cong \Omega^1(L)$, which is identified with the direct sum of the space of functions with integral 0 and that of coclosed 1-forms. It is a submanifold of the moduli space of smooth affine Legendrian embeddings of L .*

Remark 1.7. Theorem 1.6 shows the different property of the moduli space of special affine Legendrian submanifolds from that of special Legendrian submanifolds. In general, there are obstructions of special Legendrian deformations. See [9, Section 4.2].

This paper is organized as follows. In Section 2, we review the fundamental facts of Sasakian geometry. In Section 3, we review affine Lagrangian geometry and introduce its odd dimensional analogue, namely, affine Legendrian geometry. In Section 4, we compute the first variation of the ϕ -volume. In Section 5, we compute the second variation of the ϕ -volume to obtain Theorems 1.1, 1.3, 1.4 and 1.5. In Section 6, we consider the ϕ -volume in Sasaki-Einstein manifolds and introduce the notion of special affine Legendrian submanifolds. In Section 7, we study the moduli space of special affine Legendrian submanifolds and prove Theorem 1.6.

Acknowledgements: The author owes a great intellectual debt to the work of Lotay and Pacini [8], which motivates him to study the subject of this paper. He is grateful to Professor Takayuki Moriyama for letting me know his work and fruitful discussions. He also thanks the referee for useful comments on an earlier version of this paper.

2 Sasakian geometry

Definition 2.1. Let M^{2n+1} be a $(2n+1)$ -dimensional manifold. Suppose that there exist a contact form η , a Riemannian metric g , a Killing vector field ξ and a type $(1, 1)$ -tensor ϕ on M . We call (M, g, η, ξ, ϕ) a **Sasakian manifold** if we have

$$\begin{aligned}\eta(\xi) &= 1, \\ \phi^2 &= -id_{TM} + \eta \otimes \xi, \\ g(\phi(X), \phi(Y)) &= g(X, Y) - \eta(X)\eta(Y), \\ d\eta &= 2g(\cdot, \phi(\cdot)), \\ [\phi, \phi](X, Y) + d\eta(X, Y)\xi &= 0,\end{aligned}$$

where $X, Y \in TM$, $d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y])$ and $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi(X), \phi(Y)] - \phi[\phi(X), Y] - \phi[X, \phi(Y)]$.

We can also define a Sasakian manifold in terms of a Riemannian cone.

Definition 2.2. An odd dimensional Riemannian manifold (M, g) is a **Sasakian manifold** if its Riemannian cone $(C(M), \bar{g}) = (\mathbb{R}_{>0} \times M, dr^2 + r^2g)$ is a Kähler manifold with respect to some complex structure J over $C(M)$.

Here, r is a standard coordinate of $\mathbb{R}_{>0}$ and we regard r as the function on $C(M)$. We identify M with the submanifold $\{1\} \times M \subset C(M)$.

It is known that Definitions 2.1 and 2.2 are equivalent. From Definition 2.2, we see that Sasakian geometry is regarded as the odd-dimensional analogue of Kähler geometry.

Tensors in Definition 2.1 are recovered as follows. Define the vector field $\tilde{\xi}$ and the 1-form $\tilde{\eta}$ on $C(M)$ by

$$\tilde{\xi} = -J \left(r \frac{\partial}{\partial r} \right), \quad \tilde{\eta} = \frac{1}{r^2} \bar{g}(\tilde{\xi}, \cdot) = \frac{dr \circ J}{r} = -2d_J^c \log r,$$

where $d_J^c f = -df \circ J/2 = i(\bar{\partial} - \partial)f/2$ for the function f on $C(M)$. Then we have

$$\xi = \tilde{\xi}|_{r=1}, \quad \eta = \tilde{\eta}|_{r=1}.$$

By the decomposition $TM = \ker \eta \oplus \mathbb{R}\xi$, the endomorphism $\phi \in C^\infty(M, \text{End}(TM))$ is given by

$$\phi|_{\ker \eta} = J|_{\ker \eta}, \quad \phi|_{\mathbb{R}\xi} = 0.$$

The metric g on M , $J|_{\{r=1\}}$ and the Kähler form $\bar{\omega}$ of \bar{g} are described as

$$\begin{aligned} g &= \frac{1}{2} d\eta(\phi(\cdot), \cdot) + \eta \otimes \eta, \\ J|_{\{r=1\}} &= \phi - \xi \otimes dr + \frac{\partial}{\partial r} \otimes \eta, \\ \bar{\omega} &= \frac{1}{2} d(r^2 \tilde{\eta}) = -\frac{1}{2} dd_J^c r^2. \end{aligned}$$

We summarize basic equations in Sasakian geometry. See [10, Section 2].

Lemma 2.3. *Let (M, g, η, ξ, ϕ) be a $(2n+1)$ -dimensional Sasakian manifold. Then we have*

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta = g(\xi, \cdot), \quad i(\xi)d\eta = 0,$$

where $i(\cdot)$ is an inner product.

Lemma 2.4. *Let (M, g, η, ξ, ϕ) be a Sasakian manifold. Then we have*

$$\begin{aligned} \nabla_X \xi &= -\phi(X), \\ (\nabla_X \phi)(Y) &= g(X, Y)\xi - \eta(Y)X, \\ R(X, Y)\xi &= \eta(Y)X - \eta(X)Y, \\ R(X, Y)(\phi(Z)) &= \phi(R(X, Y)Z) - g(Y, Z)\phi(X) + g(\phi(X), Z)Y \\ &\quad + g(X, Z)\phi(Y) - g(\phi(Y), Z)X, \\ \text{Ric}(\xi, X) &= \begin{cases} 2n & (X = \xi) \\ 0 & (X \in \ker \eta) \end{cases} \end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$ are vector fields on M , ∇ is the Levi-Civita connection of (M, g) , R is the curvature tensor of (M, g) and Ric is the Ricci curvature tensor of (M, g) .

Note that when (M, g) is Einstein, the scalar curvature is equal to $2n(2n+1)$.

Definition 2.5. A $(2n+1)$ -dimensional Sasakian manifold $(M^{2n+1}, g, \eta, \xi, \phi)$ is called **η -Einstein** if we have

$$\text{Ric} = Ag + (2n - A)\eta \otimes \eta$$

for some $A \in \mathbb{R}$. The constant A is called the **η -Ricci constant**.

This condition is necessary to prove Theorems 1.3 and 1.4. Note that g is Einstein if $A = 2n$.

Let L be an n -dimensional manifold admitting an immersion $\iota : L \hookrightarrow M$ into a $(2n+1)$ -dimensional Sasakian manifold. The immersion ι induces the immersion

$$\bar{\iota} : C(L) = \mathbb{R}_{>0} \times L \ni (r, x) \mapsto (r, \iota(x)) \in \mathbb{R}_{>0} \times M = C(M). \quad (2.1)$$

Definition 2.6. An immersion $\iota : L \hookrightarrow M$ is **Legendrian** if $\iota^*\eta = 0$. This is equivalent to the condition that the induced immersion $\bar{\iota} : C(L) \hookrightarrow C(M)$ given by (2.1) is Lagrangian: $\bar{\iota}^*\bar{\omega} = 0$.

3 Affine Lagrangian and affine Legendrian submanifolds

First, we review affine Lagrangian geometry by [2, 8].

3.1 Affine Lagrangian submanifolds

Let (X, h, J, ω) be a real $2n$ -dimensional almost Hermitian manifold, where h is a Hermitian metric, J is an almost complex structure and ω is an associated Kähler form. Let N be an oriented n -dimensional manifold admitting an immersion $f : N \hookrightarrow X$.

Definition 3.1. An immersion f is called **affine Lagrangian** if

$$T_{f(x)}X = f_*(T_xN) \oplus Jf_*(T_xN) \quad (3.1)$$

for any $x \in N$.

Remark 3.2. If N is Lagrangian (i.e. $f^*\omega = 0$), (3.1) is an orthogonal decomposition. The affine Lagrangian condition does not require the orthogonality of the decomposition (3.1).

Remark 3.3. The affine Lagrangian condition is an open condition. The metric is not needed in the definition, and hence we can define affine Lagrangian submanifolds in an almost complex manifold.

Next, we define a J -volume introduced by Borrelli [2].

Definition 3.4. Let $f : N^n \hookrightarrow X^{2n}$ be an affine Lagrangian immersion. Define the J -**volume form** $\text{vol}_J[f]$ of f , which is the n -form on N , by

$$\text{vol}_J[f] = \rho_J[f] \text{vol}_{f^*h},$$

where vol_{f^*h} is the Riemannian volume form of f^*h and the function $\rho_J[f]$ on N is defined by

$$\rho_J[f](x) = \sqrt{\text{vol}_h(f_*e_1, \dots, f_*e_n, Jf_*e_1, \dots, Jf_*e_n)}$$

for $x \in N$ and $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_xN .

When N is compact, define the J -**volume** $\text{Vol}_J[f]$ of N by

$$\text{Vol}_J[f] = \int_N \text{vol}_J[f].$$

Remark 3.5. The definition of the J -volume form $\text{vol}_J[f]$ is independent of the choice of the Hermitian metric h . See [8, Section 3.2, 4.1]. Thus the J -volume is also defined in an almost complex manifold.

By definition, the following is easy to prove and we omit the proof.

Lemma 3.6. *We have $0 \leq \rho_J[f] \leq 1$. The equality $\rho_J[f] = 1$ holds if and only if f is Lagrangian. The equality $\rho_J[f] = 0$ holds if and only if f is partially complex, namely, f_*T_xN contains a complex line for any $x \in N$.*

Lemma 3.7. *For any diffeomorphism $\varphi \in \text{Diff}^\infty(N)$ of N , we have*

$$\text{vol}_J[f \circ \varphi] = \varphi^* \text{vol}_J[f].$$

Thus when N is compact, we obtain

$$\text{Vol}_J[f \circ \varphi] = \text{Vol}_J[f].$$

Proof. We easily see that $\rho_J[f \circ \varphi](x) = \rho_J[f](\varphi(x))$, $\text{vol}_{(f \circ \varphi)^*h} = \varphi^* \text{vol}_{f^*h}$, which imply the statement. \square

3.2 Affine Legendrian submanifolds

Next, we introduce the odd dimensional analogue of affine Lagrangian geometry, namely, affine Legendrian geometry. Let (M, g, η, ξ, ϕ) be a $(2n+1)$ -dimensional Sasakian manifold and L be an oriented n -dimensional manifold admitting an immersion $\iota : L \hookrightarrow M$.

Definition 3.8. An immersion ι is called **affine Legendrian** if

$$T_{\iota(x)}M = \iota_*(T_xL) \oplus \phi\iota_*(T_xL) \oplus \mathbb{R}\xi_{\iota(x)} \quad (3.2)$$

for any $x \in L$.

Then we can define canonical projections

$$\pi_L : T_xM \rightarrow \iota_*(T_xL), \quad \pi_\phi : T_xM \rightarrow \phi\iota_*(T_xL), \quad \pi_\xi : T_xM \rightarrow \mathbb{R}\xi_{\iota(x)}. \quad (3.3)$$

Remark 3.9. If L is Legendrian (i.e. $\iota^*\eta = 0$), (3.2) is an orthogonal decomposition. The affine Legendrian condition does not require the orthogonality of the decomposition (3.2).

Remark 3.10. We can define affine Legendrian submanifolds in an almost contact manifold. To simplify the computations, especially in Sections 4 and 5, we assume that M is Sasakian.

By definition, we easily see the following.

Remark 3.11. An immersion $\iota : L \rightarrow M$ is affine Legendrian if and only if $\bar{\iota} : C(L) \rightarrow C(M)$ given by (2.1) is affine Lagrangian.

Next, we define the ϕ -volume as an analogue of the J -volume. Recall that the Riemannian volume form $\text{vol}_{\bar{\iota}^*\bar{g}}$ of $\bar{\iota}^*\bar{g}$ on $C(L)$ and the Riemannian volume form vol_{ι^*g} of ι^*g on L are related by $\text{vol}_{\bar{\iota}^*\bar{g}} = r^n dr \wedge \text{vol}_{\iota^*g}$. As an analogue of this fact, we define the ϕ -volume.

Definition 3.12. Let $\iota : L^n \hookrightarrow M^{2n+1}$ be an affine Legendrian immersion into a Sasakian manifold. Define the ϕ -**volume form** $\text{vol}_\phi[\iota]$ of ι , which is the n -form on L , by

$$\text{vol}_J[\bar{\iota}] = r^n dr \wedge \text{vol}_\phi[\iota].$$

When L is compact, define the ϕ -**volume** $\text{Vol}_\phi[\iota]$ of L by

$$\text{Vol}_\phi[\iota] = \int_L \text{vol}_\phi[\iota].$$

The ϕ -volume form $\text{vol}_\phi[\iota]$ is described more explicitly as follows. Define the function $\rho_\phi[\iota]$ on L by

$$\begin{aligned} \rho_\phi[\iota](x) &= \rho_J[\bar{\iota}](1, x) \\ &= \sqrt{\text{vol}_g(\iota_*e_1, \dots, \iota_*e_n, -\xi, \phi\iota_*e_1, \dots, \phi\iota_*e_n)} \end{aligned}$$

for $x \in L$ and $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_xL . Then we see that

$$\text{vol}_\phi[\iota] = \rho_\phi[\iota] \text{vol}_{\iota^*g}.$$

As in the affine Lagrangian case, we easily see the following.

Remark 3.13. The definition of the ϕ -volume form $\text{vol}_\phi[\iota]$ is independent of the choice of the Sasakian metric g .

Lemma 3.14. We have $0 \leq \rho_\phi[\iota] \leq 1$. The equality $\rho_\phi[\iota] = 1$ holds if and only if ι is Legendrian. The equality $\rho_\phi[\iota] = 0$ holds if and only if for any $x \in L$, there exists $0 \neq X \in T_xL$ such that $\iota_*X, \phi\iota_*X \in \iota_*T_xL$ or ι_*X or $\phi\iota_*X$ is a multiple of $\xi_{\iota(x)}$.

Lemma 3.15. For any diffeomorphism $\varphi \in \text{Diff}^\infty(L)$ of L , we have

$$\text{vol}_\phi[\iota \circ \varphi] = \varphi^* \text{vol}_\phi[\iota].$$

Thus when L is compact, we obtain

$$\text{Vol}_\phi[\iota \circ \varphi] = \text{Vol}_\phi[\iota].$$

3.3 Geodesics and convexity

In [8, Section 3.1], the notion of geodesics in the space of affine Lagrangian submanifolds was introduced. Analogously, we define the notion of geodesics in the space of affine Legendrian submanifolds.

Let (M, g, η, ξ, ϕ) be a $(2n+1)$ -dimensional Sasakian manifold and L be an oriented n -dimensional manifold admitting an embedding $L \hookrightarrow M$. Let \mathcal{P} be the space of all affine Legendrian embeddings of L :

$$\mathcal{P} = \{\iota : L \hookrightarrow M; \iota \text{ is an affine Legendrian embedding}\}.$$

The group $\text{Diff}^\infty(L)$ of diffeomorphisms of L acts freely on \mathcal{P} on the right by reparametrizations. Set $\mathcal{A} = \mathcal{P}/\text{Diff}^\infty(L)$. Thus we can regard \mathcal{P} as a principal $\text{Diff}^\infty(L)$ -bundle over \mathcal{A} . Denote by $\pi : \mathcal{P} \rightarrow \mathcal{A}$ the canonical projection.

For each $\iota \in \mathcal{P}$, define the subspaces of $T_\iota \mathcal{P}$ by

$$V_\iota = \{\iota_* X; X \in \mathfrak{X}(L)\}, \quad H_\iota = \{\phi \iota_* Y + f\xi \circ \iota; Y \in \mathfrak{X}(L), f \in C^\infty(L)\}.$$

We easily see that $V_\iota = \ker((d\pi)_\iota : T_\iota \mathcal{P} \rightarrow T_{\pi(\iota)} \mathcal{A})$ and we have a decomposition $T_\iota \mathcal{P} = V_\iota \oplus H_\iota$. As in the proof of [8, Lemma 3.1], we see that the distribution $\iota \mapsto H_\iota$ on \mathcal{P} is $\text{Diff}^\infty(L)$ -invariant. Thus the distribution $\iota \mapsto H_\iota$ defines a connection on the principal $\text{Diff}^\infty(L)$ -bundle \mathcal{P} .

It is known that the associated vector bundle $\mathcal{P} \times_{\text{Diff}^\infty(L)} (\mathfrak{X}(L) \times C^\infty(L))$ to the standard action of $\text{Diff}^\infty(L)$ on $\mathfrak{X}(L) \times C^\infty(L)$ is isomorphic to the tangent bundle $T\mathcal{A}$:

$$\mathcal{P} \times_{\text{Diff}^\infty(L)} (\mathfrak{X}(L) \times C^\infty(L)) \cong T\mathcal{A}$$

via $[\iota, (Y, f)] \mapsto (d\pi)_\iota(\phi \iota_* Y + f\xi \circ \iota)$. Then the connection on \mathcal{P} induces a connection on $T\mathcal{A}$. We define the geodesic $\{L_t\}$ on \mathcal{A} by requiring that $\frac{dL_t}{dt}$ is parallel with respect to this connection.

Lemma 3.16. *A curve $\{L_t\} \subset \mathcal{A}$ is a geodesic if and only if there exists a curve of affine Legendrian embeddings $\{\iota_t\}$, a fixed vector field $Y \in \mathfrak{X}(L)$ and a function $f \in C^\infty(L)$ such that $\pi(\iota_t) = L_t$ and*

$$\frac{d\iota_t}{dt} = \phi(\iota_t)_* Y + f\xi \circ \iota_t.$$

This implies that $[(\iota_t)_ Y, \phi(\iota_t)_* Y + f\xi \circ \iota_t] = 0$ for all t for which $\{L_t\}$ is defined.*

Proof. Let $\{L_t\}_{t \in (a,b)} \subset \mathcal{A}$ be a geodesic and $\{x(s)\}_{s \in (c,d)}$ be an integral curve of Y on L . Then

$$f : (c, d) \times (a, b) \ni (s, t) \mapsto \iota_t(x(s)) \in M$$

is an embedded surface in M and

$$\frac{\partial f}{\partial t} = \phi(\iota_t)_* Y + f\xi \circ \iota_t, \quad \frac{\partial f}{\partial s} = (\iota_t)_* Y,$$

which imply that they commute. \square

Definition 3.17. A functional $F : \mathcal{A} \rightarrow \mathbb{R}$ is **convex** if and only if $\{F \circ L_t\}$ is a convex function in one variable for any geodesic $\{L_t\}$ in \mathcal{A} .

Remark 3.18. The existence theory of a geodesic for any $Y \in \mathfrak{X}(L)$ and $f \in C^\infty(L)$ is not known as in the case of the standard Riemannian geometry.

4 First variation of the ϕ -volume

Let $(M^{2n+1}, g, \eta, \xi, \phi)$ be a $(2n+1)$ -dimensional Sasakian manifold. Let $\iota : L^n \hookrightarrow M$ be an affine Legendrian immersion of an oriented n -dimensional manifold L . For simplicity, we identify ι_*X with X for $X \in TL$.

Fix a point $x \in L$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_x L$. Since ι is affine Legendrian, $\{e_1, \dots, e_n, \phi(e_1), \dots, \phi(e_n), \xi\}$ is the basis of $T_{\iota(x)}M$. Let $\{e^1, \dots, e^n, f^1, \dots, f^n, \eta^*\} \subset T_{\iota(x)}^*M$ be the dual basis. We easily see the following.

Lemma 4.1. *Use the notation in (3.3). We have*

$$\begin{aligned} e^i &= g(\pi_L(\cdot), e_i), & \eta^* &= g(\xi, \pi_\xi(\cdot)) = \eta(\pi_\xi(\cdot)) = \eta - \eta \circ \pi_L, \\ e^i &= f^i \circ \phi, & f^i &= -e^i \circ \phi. \end{aligned}$$

In particular, we have

$$\eta \circ \pi_L \circ \phi = -\eta^* \circ \phi.$$

Now, we compute the first variation of the ϕ -volume form. We first give all the statements in this section and then prove them.

Proposition 4.2. *Let $\iota_t : L \hookrightarrow M$ be a one-parameter family of affine Legendrian immersions satisfying $\iota_0 = \iota$. Set $\frac{\partial \iota_t}{\partial t}|_{t=0} = Z \in C^\infty(L, \iota^*TM)$. Then at $x \in L$ we have*

$$\left. \frac{\partial}{\partial t} \text{vol}_\phi[\iota_t] \right|_{t=0} = \left(\sum_{i=1}^n e^i(\nabla_{e_i} Z) - \eta^*(\phi(Z)) \right) \text{vol}_\phi[\iota].$$

Definition 4.3. Define the vector field $H_\phi \in \mathfrak{X}(L)$ on L by

$$H_\phi = -(\phi \text{tr}_L(\pi_\phi^t \nabla \pi_L^t))^\top + \xi^\top,$$

where tr_L is a trace on L , $\top : \iota^*TM \rightarrow \iota_*TL$ is the tangential projection defined by the orthogonal decomposition of ι^*TM by the metric g and

$$\pi_L^t : \iota^*TM \rightarrow (\phi(\iota_*TL) \oplus \mathbb{R}\xi \circ \iota)^\perp,$$

where $(\phi(\iota_*TL) \oplus \mathbb{R}\xi \circ \iota)^\perp$ is the orthogonal complement of $\phi(\iota_*TL) \oplus \mathbb{R}\xi \circ \iota$ with respect to g , is the transposed operator of π_L defined in (3.3) via the metric g , namely

$$g(\pi_L^t X, Y) = g(X, \pi_L Y)$$

for any $X, Y \in \iota^*TM$. Similarly, we can define transposed operators $\pi_\phi^t : \iota^*TM \rightarrow (\iota_*TL \oplus \mathbb{R}\xi \circ \iota)^\perp$ and $\pi_\xi^t : \iota^*TM \rightarrow (\iota_*TL \oplus \phi(\iota_*TL))^\perp$ of π_ϕ and π_ξ , respectively.

The vector field ϕH_ϕ is a generalization of a mean curvature vector. See Remark 4.6.

Corollary 4.4. *Let $X, Y \in \mathfrak{X}(L)$ be vector fields on L and $f \in C^\infty(L)$ be a smooth function. Then we have*

$$\sum_{i=1}^n e^i(\nabla_{e_i}(X + \phi Y + f\xi)) = \frac{\text{div}(\rho_\phi[\iota]X)}{\rho_\phi[\iota]} - g(Y, H_\phi) + \eta(Y).$$

From Proposition 4.2 and Corollary 4.4, we immediately see the following first variation formula of the ϕ -volume.

Proposition 4.5. *Use the notation of Proposition 4.2. Suppose that L is compact and $\frac{\partial \iota_t}{\partial t}|_{t=0} = Z = \phi Y + f\xi$, where $Y \in \mathfrak{X}(L)$ is a vector field on L and $f \in C^\infty(L)$ is a smooth function. Then we have*

$$\left. \frac{d}{dt} \text{Vol}_\phi[\iota_t] \right|_{t=0} = - \int_L g(Y, H_\phi) \text{vol}_\phi[l].$$

In particular, ι is a critical point for the ϕ -volume if and only if $H_\phi = 0$.

Note that f does not appear in this formula. We call an immersion ϕ -**minimal** if $H_\phi = 0$.

Remark 4.6. Suppose that L is Legendrian. Let H be the mean curvature vector of L . Then we have

$$H_\phi = -\phi H, \quad \phi H_\phi = H.$$

Proof of Proposition 4.2. Denote by $\text{vol}_\phi[\iota_t]$ the ϕ -volume of ι_t . Let $\{e_1(t), \dots, e_n(t)\}$ be an oriented orthonormal basis of $T_x L$ with respect to $\iota_t^* g$ and $\{e^1(t), \dots, e^n(t)\} \subset T_x^* L$ be the dual basis. Then we have

$$\begin{aligned} \text{vol}_\phi[\iota_t]_x &= \rho_\phi[\iota_t](x) e^1(t) \wedge \dots \wedge e^n(t), \\ \rho_\phi[\iota_t](x) &= \sqrt{(\text{vol}_g)_{\iota_t(x)}((\iota_t)_* e_1(t), \dots, (\iota_t)_* e_n(t), -\xi_{\iota_t(x)}, \phi(\iota_t)_* e_1(t), \dots, \phi(\iota_t)_* e_n(t))}. \end{aligned}$$

Since

$$(\iota_t)_*(e_1 \wedge \dots \wedge e_n) = (e^1(t) \wedge \dots \wedge e^n(t))(e_1, \dots, e_n) \cdot (\iota_t)_*(e_1(t) \wedge \dots \wedge e_n(t)),$$

it follows that

$$\text{vol}_\phi[\iota_t]_x = \rho_\phi(t)(x) \cdot \text{vol}_{\iota^* g},$$

where

$$\rho_\phi(t)(x) = \sqrt{(\text{vol}_g)_{\iota_t(x)}((\iota_t)_* e_1, \dots, (\iota_t)_* e_n, -\xi_{\iota_t(x)}, \phi(\iota_t)_* e_1, \dots, \phi(\iota_t)_* e_n)}.$$

Thus we may consider $\frac{\partial}{\partial t} \rho_\phi(t)(x)|_{t=0}$. Set

$$\nabla_Z e_i = \nabla_{\frac{\partial}{\partial t}} (\iota_t)_* e_i|_{t=0}, \quad \nabla_Z (\phi e_i) = \nabla_{\frac{\partial}{\partial t}} (\phi(\iota_t)_* e_i)|_{t=0}.$$

Since the volume form is parallel, we have

$$\begin{aligned} & \left. \frac{\partial}{\partial t} \rho_\phi(t)(x) \right|_{t=0} \\ &= \frac{\sum_{i=1}^n (\text{vol}_g)_{\iota(x)}((e_1, \dots, \nabla_Z e_i, \dots, e_n, -\xi_{\iota(x)}, \phi e_1, \dots, \phi e_n))}{2\rho_\phi[l]} \\ &+ \frac{\sum_{i=1}^n (\text{vol}_g)_{\iota(x)}(e_1, \dots, e_n, -\xi_{\iota(x)}, \phi e_1, \dots, \nabla_Z (\phi e_i), \dots, \phi e_n)}{2\rho_\phi[l]} \\ &+ \frac{(\text{vol}_g)_{\iota(x)}(e_1, \dots, e_n, -\nabla_{\frac{\partial}{\partial t}} \xi_{\iota_t(x)}|_{t=0}, \phi e_1, \dots, \phi e_n)}{2\rho_\phi[l]}. \end{aligned}$$

Using the notation at the beginning of Section 4, we have

$$\begin{aligned} & \left. \frac{\partial}{\partial t} \rho_\phi(t)(x) \right|_{t=0} \\ &= \frac{\sum_{i=1}^n e^i(\nabla_Z e_i) \rho_\phi[l]^2}{2\rho_\phi[l]} + \frac{\sum_{i=1}^n f^i(\nabla_Z(\phi e_i)) \rho_\phi[l]^2}{2\rho_\phi[l]} - \frac{\eta^*(\phi(Z)) \rho_\phi[l]^2}{2\rho_\phi[l]}. \end{aligned}$$

By Lemmas 2.4 and 4.1, it follows that

$$\begin{aligned} \nabla_Z(\phi e_i) &= g(Z, e_i)\xi - \eta(e_i)Z + \phi(\nabla_Z e_i), \\ \sum_{i=1}^n f^i(\nabla_Z(\phi e_i)) &= \sum_{i=1}^n e^i(\nabla_Z e_i) - \eta^*(\phi(Z)). \end{aligned}$$

Thus we obtain

$$\left. \frac{\partial}{\partial t} \rho_\phi(t)(x) \right|_{t=0} = \left(\sum_{i=1}^n e^i(\nabla_Z e_i) - \eta^*(\phi(Z)) \right) \rho_\phi[l].$$

Here, let (x_1, \dots, x_n) be a normal coordinate at $x \in L$ of L satisfying $e_i = \frac{\partial}{\partial x_i}$ at $x \in L$. Define the map $\tilde{\iota} : L \times (-\epsilon, \epsilon) \rightarrow M$ by $\tilde{\iota}(x, t) = \iota_t(x)$. Then $\tilde{\iota}_*(\frac{\partial}{\partial t})|_{t=0} = Z$. Since we may regard (x_1, \dots, x_n, t) as the local coordinate of $L \times (-\epsilon, \epsilon)$ near $(x, 0)$, we have $\nabla_Z e_i = \nabla_{e_i} Z$. Thus we obtain the statement. \square

Proof of Corollary 4.4. Setting $Z = X \in \mathfrak{X}(L)$ in Proposition 4.2, we have

$$\left. \frac{\partial}{\partial t} \text{vol}_\phi[l_t] \right|_{t=0} = \sum_{i=1}^n e^i(\nabla_{e_i} X) \text{vol}_\phi[l].$$

By Lemma 3.15, it follows that

$$\left. \frac{\partial}{\partial t} \text{vol}_\phi[l_t] \right|_{t=0} = L_X \text{vol}_\phi[l] = \text{div}(\rho_\phi[l]X) \text{vol}_l^* g.$$

Thus we obtain

$$\sum_{i=1}^n e^i(\nabla_{e_i} X) = \frac{\text{div}(\rho_\phi[l]X)}{\rho_\phi[l]}.$$

Using the notation of (3.3), we have

$$\begin{aligned} \sum_{i=1}^n e^i(\nabla_{e_i}(\phi Y)) &= \sum_{i=1}^n g(\pi_L(\nabla_{e_i}(\phi Y)), e_i) \\ &= - \sum_{i=1}^n g(\pi_\phi(\phi Y), \nabla_{e_i}(\pi_L^t e_i)) = g(Y, -H_\phi + \xi^\top). \end{aligned}$$

by Lemma 4.1. It is easy to show that $\sum_{i=1}^n e^i(\nabla_{e_i}(f\xi)) = 0$ and the proof is done. \square

Note that we can also prove Corollary 4.4 by a direct computation.

Proof of Remark 4.6. Since L is Legendrian, we have $\pi_L^t = \pi_L$ and $\pi_\phi^t = \pi_\phi$. Then

$$H_\phi = - \left(\phi \sum_{i=1}^n \pi_\phi \nabla_{e_i} e_i \right)^\top = - \phi \sum_{i=1}^n \pi_\phi \nabla_{e_i} e_i.$$

Let $\pi_\perp : \iota^* TM = \iota_* TL \oplus (\iota_* TL)^\perp \rightarrow (\iota_* TL)^\perp$ be the normal projection with respect to g . Then we see that

$$\begin{aligned} H &= \pi_\perp \left(\sum_{i=1}^n \nabla_{e_i} e_i \right) = \pi_\phi \left(\sum_{i=1}^n \nabla_{e_i} e_i \right) + \pi_\xi \left(\sum_{i=1}^n \nabla_{e_i} e_i \right), \\ \pi_\xi (\nabla_{e_i} e_i) &= g(\nabla_{e_i} e_i, \xi) \xi = e_i(g(e_i, \xi)) \xi + g(e_i, \phi e_i) \xi = 0, \end{aligned}$$

which implies the statement. \square

5 Second variation of the ϕ -volume

In this section, we compute the second variation of the ϕ -volume and prove Theorems 1.1, 1.3, 1.4 and 1.5. Use the notation in Section 4. First, we compute the second variation of the ϕ -volume form.

Proposition 5.1. *Let $\iota_t : L \hookrightarrow M$ be a one-parameter family of affine Legendrian immersions satisfying $\iota_0 = \iota$. Set $\frac{\partial \iota_t}{\partial t}|_{t=0} = Z \in C^\infty(L, \iota^* TM)$. Then at $x \in L$ we have*

$$\begin{aligned} \left. \frac{\partial^2}{\partial t^2} \text{vol}_\phi[l_t] \right|_{t=0} &= \left\{ -2\eta^*(\phi(Z)) \sum_i e^i(\nabla_{e_i} Z) \right. \\ &\quad - \sum_{i,j} e^i(\nabla_{e_j} Z) e^j(\nabla_{e_i} Z) + \sum_{i,j} f^i(\nabla_{e_j} Z) f^j(\nabla_{e_i} Z) \\ &\quad + \sum_i e^i(R(Z, e_i)Z + \nabla_{e_i} \nabla_Z Z) \\ &\quad - \eta(\pi_L Z) \eta^*(Z) - \eta^*(\phi(\nabla_Z Z)) - g(Z, \pi_\phi Z) \\ &\quad - 2 \sum_i f^i(Z) \eta^*(\nabla_{e_i} Z) + 2 \sum_i e^i(Z) \eta^*(\phi(\nabla_{e_i} Z)) \\ &\quad \left. + \left(\sum_i e^i(\nabla_{e_i} Z) \right)^2 \right\} \text{vol}_\phi[l]. \end{aligned}$$

where R is the curvature tensor of (M, g) .

In particular, when $Z = X \in \mathfrak{X}(L)$, we have

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \text{vol}_\phi[\iota_t]|_{t=0} &= \text{div}(\text{div}(\rho_\phi X)X) \text{vol}_{\iota^*g} \\ &= \left\{ - \sum_{i,j} e^i(\nabla_{e_j} X) e^j(\nabla_{e_i} X) + \sum_{i,j} f^i(\nabla_{e_j} X) f^j(\nabla_{e_i} X) \right. \\ &\quad + \sum_i e^i(R(X, e_i)X + \nabla_{e_i} \nabla_X X) \\ &\quad \left. + \eta^*(\phi(\nabla_X X)) + \left(\sum_i e^i(\nabla_{e_i} X) \right)^2 \right\} \text{vol}_\phi[\iota]. \end{aligned}$$

By Proposition 5.1, we obtain the second variation formula of the ϕ -volume (Theorem 1.1).

Proof of Proposition 5.1. Use the notation in the proof of Proposition 4.2. Set

$$\begin{aligned} \nabla_Z e_i &= \nabla_{\frac{\partial}{\partial t}}(\iota_t)_* e_i|_{t=0}, & \nabla_Z(\phi e_i) &= \nabla_{\frac{\partial}{\partial t}}(\phi(\iota_t)_* e_i)|_{t=0}, \\ \nabla_Z \nabla_Z e_i &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}}(\iota_t)_* e_i|_{t=0}, & \nabla_Z \nabla_Z(\phi e_i) &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}}(\phi(\iota_t)_* e_i)|_{t=0}. \end{aligned}$$

Then we have

$$\left. \frac{\partial^2}{\partial t^2} \rho_\phi^2(t) \right|_{t=0} = \sum_{i=1}^{11} h_i,$$

where

$$\begin{aligned}
h_1 &= \sum_{i \neq j} \text{vol}_g(e_1, \dots, \nabla_Z e_i, \dots, \nabla_Z e_j, \dots, e_n, -\xi, \phi e_1, \dots, \phi e_n), \\
h_2 &= \sum_{i=1}^n \text{vol}_g(e_1, \dots, \nabla_Z \nabla_Z e_i, \dots, e_n, -\xi, \phi e_1, \dots, \phi e_n), \\
h_3 &= \sum_{i=1}^n \text{vol}_g(e_1, \dots, \nabla_Z e_i, \dots, e_n, -\nabla_Z \xi, \phi e_1, \dots, \phi e_n), \\
h_4 &= \sum_{i,j=1}^n \text{vol}_g(e_1, \dots, \nabla_Z e_i, \dots, e_n, -\xi, \phi e_1, \dots, \nabla_Z(\phi e_j), \dots, \phi e_n), \\
h_5 &= \sum_{i \neq j} \text{vol}_g(e_1, \dots, e_n, -\xi, \phi e_1, \dots, \nabla_Z(\phi e_i), \dots, \nabla_Z(\phi e_j), \dots, \phi e_n), \\
h_6 &= \sum_{i=1}^n \text{vol}_g(e_1, \dots, e_n, -\xi, \phi e_1, \dots, \nabla_Z \nabla_Z(\phi e_i), \dots, \phi e_n), \\
h_7 &= \sum_{i=1}^n \text{vol}_g(e_1, \dots, e_n, -\nabla_Z \xi, \phi e_1, \dots, \nabla_Z(\phi e_i), \dots, \phi e_n), \\
h_8 &= \sum_{i,j=1}^n \text{vol}_g(e_1, \dots, \nabla_Z e_i, \dots, e_n, -\xi, \phi e_1, \dots, \nabla_Z(\phi e_j), \dots, \phi e_n), \\
h_9 &= \sum_{i=1}^n \text{vol}_g(e_1, \dots, \nabla_Z e_i, \dots, e_n, -\nabla_Z \xi, \phi e_1, \dots, \phi e_n), \\
h_{10} &= \sum_{i=1}^n \text{vol}_g(e_1, \dots, e_n, -\nabla_Z \xi, \phi e_1, \dots, \nabla_Z(\phi e_i), \dots, \phi e_n), \\
h_{11} &= \text{vol}_g(e_1, \dots, e_n, -\nabla_Z \nabla_Z \xi, \phi e_1, \dots, \phi e_n).
\end{aligned}$$

We divide h_i 's into the following four classes and compute in each class.

- class 1: h_1, h_5 ,
- class 2: h_2, h_6, h_{11} ,
- class 3: $h_3 = h_9, h_7 = h_{10}$,
- class 4: $h_4 = h_8$.

We simplify h_i 's by Lemmas 2.3, 2.4 and 4.1. First, we compute h_i 's in class 1. It is easy to see that

$$h_1 = \left\{ \left(\sum_i e^i(\nabla_Z e_i) \right)^2 - \sum_{i,j} e^i(\nabla_Z e_j) e^j(\nabla_Z e_i) \right\} \rho_\phi[l]^2.$$

Since

$$\begin{aligned}
\nabla_Z(\phi e_i) &= g(Z, e_i)\xi - \eta(e_i)Z + \phi(\nabla_Z e_i), \\
f^i(\nabla_Z(\phi e_j)) &= -\eta(e_j)f^i(Z) + e^i(\nabla_Z e_j),
\end{aligned}$$

we have

$$\begin{aligned}
h_5 &= \sum_{i \neq j} \{f^i(\nabla_Z(\phi e_i))f^j(\nabla_Z(\phi e_j)) - f^j(\nabla_Z(\phi e_i))f^i(\nabla_Z(\phi e_j))\} \rho_\phi[l]^2 \\
&= \sum_{i,j} \{-2\eta(e_i)f^i(Z)e^j(\nabla_Z e_j) + 2\eta(e_i)f^j(Z)e^i(\nabla_Z e_j)\} \rho_\phi[l]^2 + h_1 \\
&= \left\{ 2\eta(\pi_L \phi(Z)) \sum_j e^j(\nabla_Z e_j) + 2 \sum_j \eta(\pi_L \nabla_Z e_j) f^j(Z) \right\} \rho_\phi[l]^2 + h_1 \\
&= \left\{ -2\eta^*(\phi(Z)) \sum_j e^j(\nabla_Z e_j) + 2 \sum_j \eta(\pi_L \nabla_Z e_j) f^j(Z) \right\} \rho_\phi[l]^2 + h_1.
\end{aligned}$$

Next, we compute h_i 's in class 2. We easily see that

$$\begin{aligned}
h_2 &= \sum_i e^i(\nabla_Z \nabla_Z e_i) \rho_\phi[l]^2, & h_6 &= \sum_i f^i(\nabla_Z \nabla_Z(\phi e_i)) \rho_\phi[l]^2, \\
h_{11} &= \eta^*(\nabla_Z \nabla_Z \xi) \rho_\phi[l]^2.
\end{aligned}$$

Set $\nabla_Z Z = \nabla_{\frac{\partial}{\partial t}} \tilde{t}_*(\frac{\partial}{\partial t})|_{t=0}$, where \tilde{t} is given in proof of Proposition 4.5. Since

$$\begin{aligned}
\nabla_Z \nabla_Z(\phi e_i) &= Z(g(Z, e_i))\xi - g(Z, e_i)\phi(Z) + \{g(\phi(Z), e_i) - \eta(\nabla_Z e_i)\}Z \\
&\quad - \eta(e_i)\nabla_Z Z + g(Z, \nabla_Z e_i)\xi - \eta(\nabla_Z e_i)Z + \phi(\nabla_Z \nabla_Z e_i), \\
\nabla_Z \nabla_Z \xi &= -g(Z, Z)\xi + \eta(Z)Z - \phi(\nabla_Z Z),
\end{aligned}$$

we obtain

$$\begin{aligned}
h_6 &= \{-g(Z, \pi_L Z) - g(Z, \pi_\phi Z) \\
&\quad - 2 \sum_i \eta(\nabla_Z e_i) f^i(Z) - \eta^*(\phi(\nabla_Z Z))\} \rho_\phi[l]^2 + h_2, \\
h_{11} &= \{-g(Z, Z) + \eta(Z)\eta^*(Z) - \eta^*(\phi(\nabla_Z Z))\} \rho_\phi[l]^2 \\
&= \{-g(Z, \pi_L Z) - g(Z, \pi_\phi Z) - \eta^*(\phi(\nabla_Z Z))\} \rho_\phi[l]^2.
\end{aligned}$$

Compute h_i 's in class 3 to obtain

$$\begin{aligned}
h_3 &= h_9 = \left\{ - \sum_i e^i(\nabla_Z e_i) \eta^*(\phi(Z)) - \eta^*(\nabla_Z e_i) f^i(Z) \right\} \rho_\phi[l]^2 \\
h_7 &= h_{10} = \left\{ - \sum_i \eta^*(\phi(Z)) f^i(\nabla_Z(\phi e_i)) + f^i(\phi(Z)) \eta^*(\nabla_Z(\phi e_i)) \right\} \rho_\phi[l]^2 \\
&= \left\{ \sum_i \eta^*(\phi(Z)) (\eta(e_i) f^i(Z) - e^i(\nabla_Z e_i)) \right. \\
&\quad \left. + e^i(Z) (g(Z, e_i) - \eta(e_i) \eta^*(Z) + \eta^*(\phi(\nabla_Z e_i))) \right\} \rho_\phi[l]^2 \\
&= \left\{ \eta^*(\phi(Z))^2 - \eta^*(\phi(Z)) \sum_i e^i(\nabla_Z e_i) \right. \\
&\quad \left. + g(Z, \pi_L Z) - \eta(\pi_L Z) \eta^*(Z) + \sum_i e^i(Z) \eta^*(\phi(\nabla_Z e_i)) \right\} \rho_\phi[l]^2.
\end{aligned}$$

Compute h_i 's in class 4 to obtain

$$\begin{aligned}
h_4 = h_8 &= \sum_{i,j} \{e^i(\nabla_Z e_i) f^j(\nabla_Z(\phi e_j)) - f^j(\nabla_Z e_i) e^i(\nabla_Z(\phi e_j))\} \rho_\phi[l]^2 \\
&= \sum_{i,j} \{e^i(\nabla_Z e_i) (-\eta(e_j) f^j(Z) + e^j(\nabla_Z e_j)) \\
&\quad - f^j(\nabla_Z e_i) (-\eta(e_j) e^i(Z) + e^i(\phi(\nabla_Z e_j)))\} \rho_\phi[l]^2 \\
&= \left\{ -\eta^*(\phi(Z)) \sum_i e^i(\nabla_Z e_i) + \left(\sum_i e^i(\nabla_Z e_i) \right)^2 \right. \\
&\quad \left. + \sum_i \eta^*(\phi(\nabla_Z e_i)) e^i(Z) + \sum_i f^i(\nabla_Z e_j) f^j(\nabla_Z e_i) \right\} \rho_\phi[l]^2.
\end{aligned}$$

Then by a direct computation, we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} \rho_\phi(t) \Big|_{t=0} &= \frac{1}{2\rho_\phi[l]} \left(\frac{\partial^2(\rho_\phi(t)^2)}{\partial t^2} \Big|_{t=0} - 2 \left(\frac{\partial \rho_\phi(t)}{\partial t} \right)^2 \Big|_{t=0} \right) \\
&= \frac{1}{2\rho_\phi[l]} \left(\sum_{i=1}^{11} h_i - 2 \left(\sum_i e^i(\nabla_Z e_i) - \eta^*(\phi(Z)) \right)^2 \rho_\phi[l]^2 \right) \\
&= \left\{ -2\eta^*(\phi(Z)) \sum_i e^i(\nabla_Z e_i) \right. \\
&\quad - \sum_{i,j} e^i(\nabla_Z e_j) e^j(\nabla_Z e_i) + \sum_{i,j} f^i(\nabla_Z e_j) f^j(\nabla_Z e_i) \\
&\quad + \sum_i e^i(\nabla_Z \nabla_Z e_i) \\
&\quad - \eta(\pi_L Z) \eta^*(Z) - \eta^*(\phi(\nabla_Z Z)) - g(Z, \pi_\phi Z) \\
&\quad - 2 \sum_i f^i(Z) \eta^*(\nabla_Z e_i) + 2 \sum_i e^i(Z) \eta^*(\phi(\nabla_Z e_i)) \\
&\quad \left. + \left(\sum_i e^i(\nabla_Z e_i) \right)^2 \right\} \text{vol}_\phi[l].
\end{aligned}$$

Here, take the normal coordinate (x_1, \dots, x_n) at $x \in L$ in the proof of Proposition 4.2. Then we have

$$\nabla_Z e_i = \nabla_{e_i} Z, \quad \nabla_Z \nabla_Z e_i = R(Z, e_i)Z + \nabla_{e_i} \nabla_Z Z$$

at x , which give the first statement of Proposition 5.1.

When $Z = X \in \mathfrak{X}(L)$, Lemma 3.15 implies that

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} \rho_\phi[l] \Big|_{t=0} &= L_X L_X \text{vol}_\phi[l] \\
&= d(i(X) d(i(X) \rho_\phi[l] \text{vol}_{l^*g})) = \text{div}(\text{div}(\rho_\phi[l] X) X) \text{vol}_{l^*g},
\end{aligned}$$

which gives the second statement. \square

We compute the next lemma to prove Theorem 1.1.

Lemma 5.2. *Suppose that $Z = \phi Y + f\xi$, where $Y \in \mathfrak{X}(L)$ and $f \in C^\infty(L)$. Then we have*

$$\begin{aligned} \sum_{i,j} e^i(\nabla_{e_j} Z) e^j(\nabla_{e_i} Z) &= n\eta(Y)^2 + 2\eta(Y) \sum_i f^i(\nabla_{e_i} Y) + \sum_{i,j} f^i(\nabla_{e_j} Y) f^j(\nabla_{e_i} Y), \\ \sum_{i,j} f^i(\nabla_{e_j} Z) f^j(\nabla_{e_i} Z) &= nf^2 - 2f \sum_i e^i(\nabla_{e_i} Y) + \sum_{i,j} e^i(\nabla_{e_j} Y) e^j(\nabla_{e_i} Y), \end{aligned}$$

$$\sum_i e^i(R(Z, e_i)Z + R(Y, e_i)Y) = -\text{Ric}(Y, Y) + g(Y, Y) + n\eta(Y)^2 - nf^2,$$

where Ric is the Ricci curvature of (M, g) .

Proof of Lemma 5.2. The first two equations follow by the next equation:

$$\nabla_X Z = g(X, Y)\xi - \eta(Y)X + \phi(\nabla_X Y) + X(f)\xi - f\phi(X),$$

where $X \in \mathfrak{X}(L)$ is a vector field on L .

We prove the third equation. By Lemma 2.4, we see that

$$\begin{aligned} \sum_i e^i(R(Z, e_i)Z) &= \sum_i e^i(R(Z, e_i)(\phi Y + f\xi)) \\ &= \sum_i e^i(\phi(R(Z, e_i)Y) - g(e_i, Y)\phi(Z) + g(\phi Z, Y)e_i \\ &\quad + g(Z, Y)\phi(e_i) - g(\phi e_i, Y)Z + f(\eta(e_i)Z - \eta(Z)e_i)) \\ &= \sum_i e^i \circ \phi(R(Z, e_i)Y) + (-n+1)g(Y, Y) + n\eta(Y)^2 - nf^2. \end{aligned}$$

The first term is computed as

$$\sum_i e^i \circ \phi(R(Z, e_i)Y) = \sum_i g(\pi_L \phi R(Z, e_i)Y, e_i) = - \sum_i g(R(Y, \phi \pi_L^t e_i)Z, e_i),$$

$$\begin{aligned} R(Y, \phi \pi_L^t e_i)Z &= \phi(R(Y, \phi \pi_L^t e_i)Y) - g(\phi \pi_L^t e_i, Y)\phi(Y) + g(\phi Y, Y)\phi \pi_L^t e_i \\ &\quad + g(Y, Y)\phi^2 \pi_L^t e_i - g(\phi^2 \pi_L^t e_i, Y)Y + f(\eta(\phi \pi_L^t e_i)Y - \eta(Y)\phi \pi_L^t e_i) \\ &= \phi(R(Y, \phi \pi_L^t e_i)Y) - g(Y, Y)\pi_L^t e_i + g(e_i, Y)Y - f\eta(Y)\phi \pi_L^t e_i. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \sum_i e^i \circ \phi(R(Z, e_i)Y) &= \sum_i g(-\phi(R(Y, \phi \pi_L^t e_i)Y) + g(Y, Y)\pi_L^t e_i - g(e_i, Y)Y, e_i) \\ &= \sum_i g(R(Y, \phi e_i)Y, \phi \pi_L^t e_i) + (n-1)g(Y, Y) \\ &= - \sum_i g(\pi_L \phi R(Y, \phi e_i)Y, e_i) + (n-1)g(Y, Y) \\ &= \sum_i f^i(R(Y, \phi e_i)Y) + (n-1)g(Y, Y). \end{aligned}$$

Since

$$\begin{aligned}\eta^*(R(\xi, Y)Y) &= g(\pi_\xi R(\xi, Y)Y, \xi) = g(R(Y, \pi_\xi^t \xi)\xi, Y) \\ &= g(\eta(\pi_\xi^t \xi)Y - \eta(Y)\pi_\xi^t \xi, Y) = g(Y, Y),\end{aligned}$$

it follows that

$$\begin{aligned}& \sum_i e^i(R(Z, e_i)Z + R(Y, e_i)Y) \\ &= - \sum_i (f^i(R(\phi e_i, Y)Y) + e^i(R(e_i, Y)Y)) + (n-1)g(Y, Y) \\ & \quad + (-n+1)g(Y, Y) + n\eta(Y)^2 - nf^2 \\ &= -\text{Ric}(Y, Y) + g(Y, Y) + n\eta(Y)^2 - nf^2.\end{aligned}$$

□

Proof of Theorem 1.1. Set $Z = \phi Y + f\xi$. By Proposition 5.1 and Lemma 5.2, we have

$$\begin{aligned}\left. \frac{\partial^2}{\partial t^2} \text{vol}_\phi[l_t] \right|_{t=0} &= \rho_\phi[l] \left\{ -2\eta(Y) \sum_i e^i(\nabla_{e_i} Z) \right. \\ & \quad - \sum_{i,j} e^i(\nabla_{e_j} Z) e^j(\nabla_{e_i} Z) + \sum_{i,j} f^i(\nabla_{e_j} Z) f^j(\nabla_{e_i} Z) \\ & \quad + \sum_i e^i(R(Z, e_i)Z + \nabla_{e_i} \nabla_Z Z) + \left(\sum_i e^i(\nabla_{e_i} Z) \right)^2 \\ & \quad \left. - \eta^*(\phi(\nabla_Z Z)) - g(Z, \phi Y) - 2\eta^*(\nabla_Y Z) \right\} \\ &= -\text{div}(\text{div}(\rho_\phi[l]Y)Y) \\ & \quad + \rho_\phi[l] \left\{ -2\eta(Y) \sum_i e^i(\nabla_{e_i} Z) - 2\eta(Y) \sum_i f^i(\nabla_{e_i} Y) \right. \\ & \quad - 2f \sum_i e^i(\nabla_{e_i} Y) - \text{Ric}(Y, Y) + \sum_i e^i(\nabla_{e_i} (\nabla_Z Z + \nabla_Y Y)) \\ & \quad + \left(\sum_i e^i(\nabla_{e_i} Z) \right)^2 + \left(\sum_i e^i(\nabla_{e_i} Y) \right)^2 \\ & \quad \left. - \eta^*(\phi(\nabla_Z Z)) + \eta^*(\phi(\nabla_Y Y)) - 2\eta^*(\nabla_Y Z) + \eta(Y)^2 \right\}.\end{aligned}$$

From Corollary 4.4, we have

$$\sum_i e^i(\nabla_{e_i} Z) = -g(Y, H_\phi) + \eta(Y), \quad \sum_i f^i(\nabla_{e_i} Y) = g(Y, H_\phi) - (n+1)\eta(Y),$$

which imply that

$$-2\eta(Y) \sum_i e^i(\nabla_{e_i} Z) - 2\eta(Y) \sum_i f^i(\nabla_{e_i} Y) = 2n\eta(Y)^2. \quad (5.1)$$

Since we know that

$$\begin{aligned}\nabla_Z Z &= -\eta(Y)Z + \phi(\nabla_Z Y) + fY + Z(f)\xi, \\ \nabla_Y Y &= -\phi\nabla_Y(\phi Y) + \eta(\nabla_Y Y)\xi - \eta(Y)\phi(Y), \\ \nabla_Z Z + \nabla_Y Y &= \phi([Z, Y]) - 2\eta(Y)\phi(Y) + 2fY + (Z(f) + \eta(\nabla_Y Y) - 2f\eta(Y))\xi,\end{aligned}$$

we see by Corollary 4.4 that

$$-\eta^*(\phi(\nabla_Z Z)) = \eta(Y)^2 - \eta(\pi_L(\nabla_Z Y)), \quad \eta^*(\phi(\nabla_Y Y)) = -\eta(\pi_L(\nabla_Y Z)) - \eta(Y)^2.$$

Using the equation $\eta(\nabla_Y Z) = Y(f) + g(Y, Y) - \eta(Y)^2$, it follows that

$$\begin{aligned}-\eta^*(\phi(\nabla_Z Z)) + \eta^*(\phi(\nabla_Y Y)) - 2\eta^*(\nabla_Y Z) \\ = \eta(\pi_L[Y, Z]) - 2Y(f) - 2g(Y, Y) + 2\eta(Y)^2.\end{aligned}\tag{5.2}$$

We can also compute

$$\begin{aligned}\sum_i e^i(\nabla_{e_i}(\nabla_Z Z + \nabla_Y Y)) &= \frac{\operatorname{div}(\rho_\phi[\iota](\pi_L\phi([Z, Y]) + 2fY))}{\rho_\phi[\iota]} \\ &\quad + g(-\pi_L[Z, Y] + 2\eta(Y)Y, H_\phi) \\ &\quad + \eta(\pi_L[Z, Y]) - 2\eta(Y)^2.\end{aligned}\tag{5.3}$$

Note that

$$f\operatorname{div}(\rho_\phi[\iota]Y) + \rho_\phi[\iota]Y(f) = \operatorname{div}(f\rho_\phi[\iota]Y).\tag{5.4}$$

Then by (5.1), (5.2), (5.3), (5.4) and Corollary 4.4, we obtain

$$\begin{aligned}\left.\frac{\partial^2}{\partial t^2}\rho_\phi(t)\right|_{t=0} &= -\operatorname{div}(\operatorname{div}(\rho_\phi[\iota]Y)Y) + \operatorname{div}(\rho_\phi[\iota](\pi_L\phi([Z, Y]) + 2fY)) - 2\operatorname{div}(f\rho_\phi[\iota]Y) \\ &\quad + \rho_\phi[\iota]\left\{(2n+2)\eta(Y)^2 - \operatorname{Ric}(Y, Y) - 2g(Y, Y)\right. \\ &\quad \left.- g(\pi_L[Z, Y], H_\phi) + g(Y, H_\phi)^2 + \frac{(\operatorname{div}(\rho_\phi[\iota]Y))^2}{\rho_\phi[\iota]^2}\right\},\end{aligned}$$

which implies Theorem 1.1. \square

We investigate the relation of Theorem 1.1 and the previous works. Define the standard Riemannian volume of ι by $\operatorname{Vol}[\iota] = \int_L \operatorname{vol}_{\iota^*g}$.

Remark 5.3. We call a Legendrian immersion ι Legendrian-minimal Legendrian if it is a critical point of the standard Riemannian volume functional under Legendrian variations.

Suppose that ι is Legendrian-minimal Legendrian and all of ι_t 's are Legendrian in Theorem 1.1. Then for any t , the ϕ -volume agrees with the standard Riemannian volume and the second variation formula of Theorem 1.1 is given by

$$\begin{aligned}\left.\frac{d^2}{dt^2}\operatorname{Vol}[\iota_t]\right|_{t=0} &= \int_L \left(\frac{1}{4}(\Delta f)^2 - 2g(Y, Y) - \operatorname{Ric}(\phi Y, \phi Y)\right. \\ &\quad \left.- 2g(\nabla_Y Y, H) + g(Y, H)^2\right) \operatorname{vol}_{\iota^*g},\end{aligned}$$

where H is the mean curvature vector of L and Δ is the Laplacian acting on $C^\infty(L)$. This formula agrees with [7, Theorem 1.1]. Thus when ι is minimal Legendrian and all of ι_t 's are Legendrian, it agrees with [10, Theorem 1.1].

Proof. Since ι is Legendrian, we see that

$$\eta(Y) = 0, \quad \rho_\phi[\iota] = 1, \quad H_\phi = -\phi H, \quad (5.5)$$

by Lemma 3.14 and Remark 4.6. Since all of ι_t 's are Legendrian, we have $L_Z\eta = 2g(Y, \cdot) + df = 0$, which implies that

$$\operatorname{div}(Y) = \frac{1}{2}\Delta f. \quad (5.6)$$

A direct computation gives

$$\begin{aligned} \operatorname{Ric}(Y, Y) &= \operatorname{Ric}(\phi Y, \phi Y), \\ -g(\pi_L[Z, Y], H_\phi) &= g(\nabla_Z Y - \nabla_Y Z, \phi H), \\ g(\nabla_Z Y, \phi H) &= -g(\nabla_Z Z, H) \\ g(\nabla_Y Z, \phi H) &= g(\nabla_Y Y, H). \end{aligned} \quad (5.7)$$

By [7, Lemma 4.1], we have

$$\int_L g(\nabla_Z Z, H) \operatorname{vol}_{\iota^*g} = \int_L g(\nabla_Y Y, H) \operatorname{vol}_{\iota^*g}, \quad (5.8)$$

where we use an integration by parts argument and the Legendrian-minimality of ι , which is equivalent to $\operatorname{div}(\phi H) = 0$ [7, Theorem 3.6]. Then we obtain the statement by (5.5), (5.6), (5.7) and (5.8). \square

Now we prove Theorems 1.3, 1.4 and 1.5.

Proof of Theorem 1.3. Let $\iota : L^n \hookrightarrow M^{2n+1}$ be a ϕ -minimal affine Legendrian submanifold. By definition, we have $H_\phi = 0$. Since M^{2n+1} is a η -Einstein Sasakian manifold with the η -Ricci constant A , we see from Definition 2.5 that

$$(2n+2)\eta(Y)^2 - 2g(Y, Y) - \operatorname{Ric}(Y, Y) = (A+2)(\eta(Y)^2 - g(Y, Y)) \quad (5.9)$$

for $Y \in TL$. By the third equation of Definition 2.1, we have $\eta(Y)^2 - g(Y, Y) \leq 0$. Then Theorem 1.1 implies Theorem 1.3. \square

Proof of Theorem 1.4. Recall Lemma 3.16. Let $\{L_t\} \subset \mathcal{A}$ be a geodesic. Then there exists a curve of affine Legendrian embeddings $\{\iota_t\}$, a fixed vector field $Y \in \mathfrak{X}(L)$ and a function $f \in C^\infty(L)$ such that $\pi(\iota_t) = L_t$,

$$\frac{d\iota_t}{dt} = \phi(\iota_t)_* Y + f\xi \circ \iota_t \quad \text{and} \quad [(\iota_t)_* Y, \phi(\iota_t)_* Y + f\xi \circ \iota_t] = 0. \quad (5.10)$$

Then by Theorem 1.1, (5.9) and (5.10), we obtain Theorem 1.4. \square

Proof of Theorem 1.5. Let $\iota : L^n \hookrightarrow M^{2n+1}$ be a ϕ -minimal affine Legendrian submanifold. Since M^{2n+1} is a η -Einstein Sasakian manifold with the η -Ricci constant $A > -2$, we have

$$(2n+2)\eta(Y)^2 - 2g(Y, Y) - \operatorname{Ric}(Y, Y) = (A+2)(\eta(Y)^2 - g(Y, Y)) < 0$$

for any $0 \neq Y \in TL$ by the third equation of Definition 2.1 and Definition 3.8. Take a 1-form $0 \neq \alpha \in \Omega^1(L)$ such that $d^*\alpha = 0$. For example, set $\alpha = d^*\beta$ for a 2-form β . Define the vector field $Y \in \mathfrak{X}(L)$ on L via

$$\iota^*g(\rho_\phi[l]Y, \cdot) = \alpha.$$

Then we easily see that $\operatorname{div}(\rho_\phi[l]Y) = -d^*\alpha = 0$. By Theorem 1.1, the second variation of the ϕ -volume for this Y is given by

$$\frac{d^2}{dt^2} \int_L \operatorname{vol}_\phi[l_t] \Big|_{t=0} = \int_L ((A+2)(\eta(Y)^2 - g(Y, Y))) \operatorname{vol}_\phi[l] < 0,$$

which implies that $\iota : L \hookrightarrow M$ is not ϕ -stable. \square

6 ϕ -volume in Sasaki-Einstein manifolds

6.1 J -volume in Calabi-Yau manifolds

Definition 6.1. Let (X, h, J, ω) be a real $2n$ -dimensional Kähler manifold with a Kähler metric h , a complex structure J and an associated Kähler form ω . Suppose that there exists a nowhere vanishing holomorphic $(n, 0)$ -form Ω on X satisfying

$$\omega^n/n! = (-1)^{n(n-1)/2} (i/2)^n \Omega \wedge \bar{\Omega}, \quad (6.1)$$

Then a quintuple $(X, h, J, \omega, \Omega)$ is called a **Calabi-Yau manifold**.

Remark 6.2. The condition (6.1) implies that h is Ricci-flat and Ω is parallel with respect to the Levi-Civita connection of h .

In Section 6.1, we suppose that $(X, h, J, \omega, \Omega)$ is a real $2n$ -dimensional Calabi-Yau manifold.

6.1.1 Special Lagrangian geometry

Define the **Lagrangian angle** $\theta_N : N \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ of a Lagrangian immersion $f : N \hookrightarrow X$ by

$$f^*\Omega = e^{i\theta_N} \operatorname{vol}_{f^*h}.$$

This is well-defined because $|f^*\Omega(e_1, \dots, e_n)| = 1$ for any orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x N$ for $x \in N$, which is proved in [6, Theorem III.1.7]. It also implies that $\operatorname{Re}(e^{i\theta} \Omega)$ defines a calibration on X for any $\theta \in \mathbb{R}$.

A Lagrangian immersion $f : N \hookrightarrow X$ is called **special Lagrangian** if $f^*\operatorname{Re}\Omega = \operatorname{vol}_{f^*h}$, namely, the Lagrangian angle is 0.

Minimal Lagrangian submanifolds are characterized in terms of special Lagrangian submanifolds as follows. For example, see [8, Lemma 8.1]

Lemma 6.3. *Let $f : N \hookrightarrow X$ be an immersion of an oriented connected n -dimensional manifold N . The following are equivalent.*

- (a) $f^*\operatorname{Re}(e^{i\theta}\Omega) = \operatorname{vol}_{f^*h}$ for some $\theta \in \mathbb{R}$;
- (b) $f^*\omega = 0$ and $f^*\operatorname{Im}(e^{i\theta}\Omega) = 0$ for some $\theta \in \mathbb{R}$;
- (c) $f^*\omega = 0$ and the Lagrangian angle θ_N is constant;
- (d) $f : N \hookrightarrow X$ is minimal Lagrangian.

6.1.2 Special affine Lagrangian geometry

By using a J -volume, we can generalize the notion of calibrations.

Lemma 6.4 ([8, Lemma 8.2]). *Let $f : N \hookrightarrow X$ be an affine Lagrangian immersion of an oriented n -dimensional manifold N . Then we have*

$$f^* \operatorname{Re} \Omega \leq \operatorname{vol}_J[f] \leq \operatorname{vol}_{f^*h}.$$

The equality holds

- in the first relation if and only if $f^* \operatorname{Im} \Omega = 0$ and $f^* \operatorname{Re} \Omega > 0$,
- and in the second relation if and only if f is Lagrangian.

Following [8, Section 7.1], define the **affine Lagrangian angle** $\theta_N : N \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ of an affine Lagrangian immersion $f : N \hookrightarrow X$ by

$$f^* \Omega = e^{i\theta_N} \operatorname{vol}_J[f].$$

This is well-defined because

$$|f^* \Omega(e_1, \dots, e_n)| = \rho_J[f] \quad (6.2)$$

for any orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x N$ for $x \in N$. The equation (6.2) is proved in [8, Lemma 7.2]. We can also prove this directly by a pointwise calculation.

We call an affine Lagrangian immersion $f : N \hookrightarrow X$ **special affine Lagrangian** if $f^* \operatorname{Re} \Omega = \operatorname{vol}_J[f]$, namely, the affine Lagrangian angle is 0.

Remark 6.5. When $f : N \hookrightarrow X$ is Lagrangian, we have $\operatorname{vol}_J[f] = \operatorname{vol}_{f^*h}$ by Lemma 3.6. Then the affine Lagrangian angle agrees with the standard Lagrangian angle.

We have an analogue of Lemma 6.3 given in [8, Lemma 8.3].

Lemma 6.6. *Let $f : N \hookrightarrow X$ be an affine Lagrangian immersion of an oriented connected real n -dimensional manifold N . The following are equivalent.*

- (a) $f^* \operatorname{Re}(e^{i\theta} \Omega) = \operatorname{vol}_J[f]$ for some $\theta \in \mathbb{R}$;
- (b) $f^* \operatorname{Im}(e^{i\theta} \Omega) = 0$ for some $\theta \in \mathbb{R}$;
- (c) the affine Lagrangian angle θ_N is constant;
- (d) $f : N \hookrightarrow X$ is a critical point for the J -volume.

Proof. Define $H_J \in C^\infty(N, Jf_*TN)$ by

$$H_J = -J((\operatorname{tr}_N(\pi_J^t \nabla^X \pi_N^t))^\top), \quad (6.3)$$

where ∇^X is the Levi-Civita connection of h , $\top : \iota^*TX \rightarrow \iota_*TN$ is the tangential projection defined by h , and π_N^t and π_J^t are transposed operators of the canonical projections of $\pi_N : \iota^*TX \rightarrow \iota_*TN$ and $\pi_J : \iota^*TX \rightarrow J\iota_*TN$ via the decomposition $\iota^*TX = \iota_*TN \oplus J\iota_*TN$, respectively.

By [8, Proposition 5.2], $f : N \hookrightarrow X$ is a critical point for the J -volume if and only if $H_J = 0$. By [8, Corollary 7.4], we have

$$H_J = J(d\theta_N)^\sharp, \quad (6.4)$$

where \sharp is the metric dual with respect to f^*h . Then by (6.2) and (6.4), we see the equivalence. \square

6.2 ϕ -volume in Sasaki-Einstein manifolds

The odd dimensional analogue of a Calabi-Yau manifold is a Sasaki-Einstein manifold. The following is a well-known fact. For example, see [3, Lemma 11.1.5].

Lemma 6.7. *Let (M, g, η, ξ, ϕ) be a $(2n+1)$ -dimensional Sasakian manifold. If g is Einstein, a cone $(C(M), \bar{g})$ is Ricci-flat.*

Thus the canonical bundle of $C(M)$ is diffeomorphically trivial. In addition, suppose that the cone $C(M)$ is a Calabi-Yau manifold, namely, there exists a nowhere vanishing holomorphic $(n+1, 0)$ -form Ω on $C(M)$ such that

$$\bar{\omega}^{n+1}/(n+1)! = (-1)^{n(n+1)/2} (i/2)^{n+1} \Omega \wedge \bar{\Omega}, \quad (6.5)$$

where $\bar{\omega} = \bar{g}(J, \cdot)$ is the associated Kähler form on $C(M)$. Then the canonical bundle of $C(M)$ is holomorphically trivial.

Lemma 6.8 ([3, Corollary 11.1.8]). *If M is a compact simply-connected Sasaki-Einstein manifold, $C(M)$ is a Calabi-Yau manifold.*

Remark 6.9. The holomorphic volume form Ω is not unique. For any $\theta \in \mathbb{R}$, $e^{i\theta}\Omega$ also satisfies (6.5).

In Section 6.2, we suppose that M is a $(2n+1)$ -dimensional Sasaki-Einstein manifold with a Calabi-Yau structure $(\bar{g}, J, \bar{\omega}, \Omega)$ on $C(M)$.

Define a complex valued n -form on M by

$$\psi = u^* \left(i \left(r \frac{\partial}{\partial r} \right) \Omega \right),$$

where $u : M = \{1\} \times M \hookrightarrow C(M)$ is an inclusion. Note that we can recover Ω from ψ via

$$\Omega = (dr - i\eta) \wedge r^n \psi. \quad (6.6)$$

6.2.1 Special Legendrian geometry

Define the **Legendrian angle** $\theta_L : L \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ of a Legendrian immersion $\iota : L \hookrightarrow M$ by

$$\iota^* \psi = e^{i\theta_L} \text{vol}_{\iota^* g}.$$

Note that the Legendrian angle θ_L of a Legendrian immersion $\iota : L \hookrightarrow M$ agrees with the Lagrangian angle of the induced Lagrangian immersion $\bar{\iota} : C(L) \hookrightarrow C(M)$ given by (2.1).

Definition 6.10. Let L be an oriented n -dimensional manifold admitting an immersion $\iota : L \hookrightarrow M$. An immersion $\iota : L \hookrightarrow M$ is called **special Legendrian** if $\iota^* \text{Re} \psi = \text{vol}_{\iota^* g}$. This is equivalent to the condition that the induced immersion $\bar{\iota} : C(L) \hookrightarrow C(M)$ given by (2.1) is special Lagrangian.

We have an analogue of Lemmas 6.3 and 6.6. This is a direct consequence of Lemma 6.3. Note that ι is minimal if and only if $\bar{\iota}$ is minimal.

Lemma 6.11. *Let $\iota : L \hookrightarrow M$ be an immersion of an oriented connected n -dimensional manifold L . The following are equivalent.*

- (a) $\iota^* \text{Re}(e^{i\theta} \psi) = \text{vol}_{\iota^* g}$ for some $\theta \in \mathbb{R}$;
- (b) $\iota^* \eta = 0$ and $\iota^* \text{Im}(e^{i\theta} \psi) = 0$ for some $\theta \in \mathbb{R}$;
- (c) $\iota^* \eta = 0$ and the Legendrian angle θ_L is constant;
- (d) $\iota : L \hookrightarrow M$ is minimal Legendrian.

6.2.2 Special affine Legendrian geometry

From Lemma 6.4, we immediately see the following.

Lemma 6.12. *Let $\iota : L \hookrightarrow M$ be an affine Legendrian immersion of an oriented n -dimensional manifold L . Then we have*

$$\iota^* \text{Re} \psi \leq \text{vol}_\phi[\iota] \leq \text{vol}_{\iota^* g}.$$

The equality holds

- in the first relation if and only if $\iota^* \text{Im} \psi = 0$ and $\iota^* \text{Re} \psi > 0$,
- and in the second relation if and only if ι is Legendrian.

Define the **affine Legendrian angle** $\theta_L : L \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ of an affine Legendrian immersion $\iota : L \hookrightarrow M$ by

$$\iota^* \psi = e^{i\theta_L} \text{vol}_\phi[\iota].$$

This is well-defined by (6.2). Note that the affine Legendrian angle θ_L of an affine Legendrian immersion $\iota : L \hookrightarrow M$ agrees with the affine Lagrangian angle of the induced affine Lagrangian immersion $\bar{\iota} : C(L) \hookrightarrow C(M)$ given by (2.1).

Definition 6.13. Let L be an oriented n -dimensional manifold admitting an immersion $\iota : L \hookrightarrow M$. An immersion $\iota : L \hookrightarrow M$ is called **special affine Legendrian** if $\iota^* \text{Re} \psi = \text{vol}_\phi[\iota]$. This is equivalent to the condition that the induced immersion $\bar{\iota} : C(L) \hookrightarrow C(M)$ given by (2.1) is special affine Lagrangian.

It is natural to expect an analogue of Lemmas 6.3, 6.6 and 6.11. By Lemma 6.12, we immediately see that the following three conditions are equivalent.

- $\iota^* \text{Re}(e^{i\theta} \psi) = \text{vol}_\phi[\iota]$ for some $\theta \in \mathbb{R}$;
- $\iota^* \text{Im}(e^{i\theta} \psi) = 0$ for some $\theta \in \mathbb{R}$;
- the affine Legendrian angle θ_L is constant.

However, in the affine Legendrian setting, each of these conditions is not equivalent to saying that ι is a critical point for the ϕ -volume. In fact, we have the following.

Proposition 6.14. *Let $\iota : L \hookrightarrow M$ be an affine Legendrian immersion of an oriented connected n -dimensional manifold L . We have*

$$(d\theta_L)^\sharp = -(n+1)\xi^\top + H_\phi,$$

where \sharp is the metric dual with respect to ι^*g on L , $\top : \iota^*TM \rightarrow TL$ is the tangential projection defined by the orthogonal decomposition of ι^*TM by the metric g and H_ϕ is given in Definition 4.3.

Thus an analogue of Lemmas 6.3, 6.6 and 6.11 holds if an affine Legendrian immersion $\iota : L \hookrightarrow M$ is Legendrian. It may be necessary to modify the notion of the ϕ -volume to hold an analogue of Lemmas 6.3, 6.6 and 6.11 or to consider what the critical points of the ϕ -volume which are not minimal Legendrian are.

Lemma 6.15. *Let $H_J \in C^\infty(C(L), J\iota_*TC(L))$ be defined by (6.3). Then we have*

$$(JH_J)|_{r=1} = (n+1)\xi^\top - H_\phi.$$

Proof. By (6.3), we have for any vector field Y on $C(L)$

$$\begin{aligned} \iota^*\bar{g}(Y, JH_J) &= \sum_{i=1}^n \iota^*\bar{g}\left(\pi_{C(L)}(\bar{\nabla}_{\frac{e_i}{r}}(JY)), \frac{e_i}{r}\right) + \iota^*\bar{g}\left(\pi_{C(L)}(\bar{\nabla}_{\frac{\partial}{\partial r}}(JY)), \frac{\partial}{\partial r}\right) \\ &= \sum_{i=1}^n \iota^*\bar{g}\left(\pi_{C(L)}(\bar{\nabla}_{\frac{e_i}{r}}(JY)), \frac{e_i}{r}\right), \end{aligned}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame of TL with respect to ι^*g . Using the notation in Section 4, we have for any vector field Y on L

$$\begin{aligned} \bar{\nabla}_{e_i}(JY) &= J\left(\nabla_{e_i}Y - \frac{\bar{g}(e_i, Y)}{r^2} \cdot r \frac{\partial}{\partial r}\right), \\ \nabla_{e_i}Y &= \sum_{j=1}^n e^j(\nabla_{e_i}Y)e_j + \sum_{j=1}^n f^j(\nabla_{e_i}Y)\phi e_j + \eta^*(\nabla_{e_i}Y)\xi \\ &= \sum_{j=1}^n e^j(\nabla_{e_i}Y)e_j + \sum_{j=1}^n f^j(\nabla_{e_i}Y)\left(Je_j - \eta(e_j)r \frac{\partial}{\partial r}\right) + \eta^*(\nabla_{e_i}Y)\xi. \end{aligned}$$

Hence we have at the point of $\{r=1\}$

$$\begin{aligned} \sum_{i=1}^n \iota^*\bar{g}(\pi_{C(L)}(\bar{\nabla}_{e_i}(JY)), e_i) &= -\sum_{i=1}^n f^i(\nabla_{e_i}Y) \\ &= \sum_{i=1}^n e^i(\phi(\nabla_{e_i}Y)) \\ &= \sum_{i=1}^n e^i(\nabla_{e_i}(\phi Y) - (\nabla_{e_i}\phi)(Y)) \\ &= \sum_{i=1}^n e^i(\nabla_{e_i}(\phi Y)) + n\eta(Y). \end{aligned}$$

Since we know that $\sum_{i=1}^n e^i(\nabla_{e_i}(\phi Y)) = -g(Y, H_\phi) + \eta(Y)$ by Corollary 4.4, the proof is done. \square

Proof of Proposition 6.14. By (6.4), we have

$$d\theta_{C(L)} = -\bar{\iota}^* \bar{g}(JH_J, \cdot),$$

where $\theta_{C(L)}$ is the affine Lagrangian angle of $\bar{\iota} : C(L) \hookrightarrow C(M)$ given by (2.1). Since we know that $u^* \theta_{C(L)} = \theta_L$ for the inclusion $u : L \hookrightarrow C(L)$, Lemma 6.15 implies the statement. \square

Remark 6.16. We can also prove Proposition 6.14 by using the tensors on L . We give an outline of the proof. Define the 1-form ξ_ϕ on L by $u^* \xi_J$: the pullback of the Maslov form ξ_J of $C(L)$ defined in [8, Section 3.2] by $u : L \hookrightarrow C(L)$. Define the complex valued n -form ψ_L on L by $\psi_L = u^*(i(\partial/\partial r)\Omega_{C(L)})$, where $\Omega_{C(L)}$ is the canonical section of the canonical bundle of $C(L)$ defined in [8, Section 3.2].

Then as in [8, Lemma 7.2], we have $\psi = e^{i\theta_L} \psi_L$. By a direct computation, we have

$$\xi_\phi = -\sum_{i=1}^n f^i(\nabla e_i), \quad \xi_\phi^\# = (n+1)\xi^\top - H_\phi.$$

Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections of \bar{g} and g , respectively. By the equations $\bar{\nabla}\Omega = 0$ and (6.6), we deduce that

$$\nabla\psi = -i\eta \wedge \psi.$$

By the equations $\bar{\nabla}\Omega_L = i\xi_J \otimes \Omega_{C(L)}$ and $\Omega_{C(L)} = (dr - i\eta) \wedge r^n \psi_L$, we deduce that

$$\nabla\psi_L = i(-\eta \wedge \psi_L + \xi_\phi \otimes \psi_L).$$

Then we obtain $\xi_\phi = -d\theta_L$ from $\psi = e^{i\theta_L} \psi_L$.

7 Moduli space of the special affine Legendrian submanifolds

In this section, we prove Theorem 1.6. First, we study the moduli space of submanifolds characterized by differential forms following [9] to obtain Proposition 7.2. As a corollary of Proposition 7.2, we prove Theorem 1.6.

Let (M, g) be a Riemannian manifold and L be a compact connected manifold admitting an embedding into M . Denote by $C_{emb}^\infty(L, M)$ be the set of all embeddings from L to M :

$$C_{emb}^\infty(L, M) = \{\iota : L \hookrightarrow M; \iota \text{ is an embedding}\}.$$

Set $\mathcal{M}(L, M) = C_{emb}^\infty(L, M)/\text{Diff}^\infty(L)$, where $\text{Diff}^\infty(L)$ is a C^∞ diffeomorphism group of L .

By [11, Theorem 3.3], $\mathcal{M}(L, M)$ is a smooth Fréchet manifold modeled on the Fréchet vector space $C^\infty(L, \mathcal{N}_\iota)$ for $\iota \in C_{emb}^\infty(L, M)$, where \mathcal{N}_ι is any vector bundle transversal to ι and $C^\infty(L, \mathcal{N}_\iota)$ is the space of all sections of $\mathcal{N}_\iota \rightarrow L$.

Now we choose a system $\Phi = (\varphi_1, \dots, \varphi_m) \in \oplus_{i=1}^m \Omega^{k_i}(M)$ of smooth differential forms on M . These forms are not necessarily closed.

Definition 7.1. The embedding $\iota \in C_{emb}^\infty(L, M)$ is called a Φ -embedding if

$$\iota^* \Phi = (\iota^* \varphi_1, \dots, \iota^* \varphi_m) = 0.$$

Define the moduli space $\mathcal{M}_L(\Phi)$ of Φ -embeddings of L by

$$\mathcal{M}_L(\Phi) = \{\iota \in C_{emb}^\infty(L, M); \iota^* \Phi = 0\} / \text{Diff}^\infty(L).$$

We want to study the structure of $\mathcal{M}_L(\Phi)$.

Fix $\iota \in C_{emb}^\infty(L, M)$ satisfying $\iota^* \Phi = 0$ and a vector bundle $\mathcal{N}_\iota \rightarrow L$ which is transversal to ι . Set

$$V_1 = C^\infty(L, \mathcal{N}_\iota), \quad V_2 = \oplus_{i=1}^m \Omega^{k_i}(L) = C^\infty(L, \oplus_{i=1}^m \wedge^{k_i} T^*L).$$

By the tubular neighborhood theorem there exists a neighborhood of L in M which is identified with an open neighborhood $\mathcal{U} \subset \mathcal{N}_\iota$ of the zero section by the exponential map. Set

$$U = \{v \in V_1; v_x \in \mathcal{U} \text{ for any } x \in L\}.$$

The exponential map induces the embedding $\exp_v : L \hookrightarrow M$ by $\exp_v(x) = \exp_x(v_x)$ for $v \in U$ and $x \in L$. Define the first order differential operator $F : U \rightarrow V_2$ by

$$F(v) = \exp_v^* \Phi = (\exp_v^* \varphi_1, \dots, \exp_v^* \varphi_m).$$

Then $\exp_v : L \hookrightarrow M$ is Φ -embedding if and only if $F(v) = 0$. Thus a neighborhood of $[\iota]$ in $\mathcal{M}_L(\Phi)$ is identified with that of 0 in $F^{-1}(0)$ (in the C^1 sense). Let D_1 be the linearization of F at 0:

$$D_1 = (dF)_0 : V_1 \rightarrow V_2.$$

First, we prove the following, which is a slight generalization of [9, Proposition 2.2]. It will be useful to see whether the moduli space of submanifolds characterized by some differential forms is smooth. We use the notion of a Fréchet manifold given in [5].

Proposition 7.2. *Suppose that there exist a vector bundle $E \rightarrow L$ and a first order differential operator $D_2 : V_2 \rightarrow V_3$, where $V_3 = C^\infty(L, E)$ is a space of smooth sections of $E \rightarrow L$, such that*

$$V_1 \xrightarrow{D_1} V_2 \xrightarrow{D_2} V_3$$

is a differential complex. Namely, $D_2 \circ D_1 = 0$. Denote by $D_i^ : V_{i+1} \rightarrow V_i$ the formal adjoint operator of D_i .*

1. *Suppose that $P_2 = D_1 D_1^* + D_2^* D_2 : V_2 \rightarrow V_2$ is elliptic and $\text{Im}(F) \subset \text{Im}(D_1)$. Then around $[\iota]$, the moduli space $\mathcal{M}_L(\Phi)$ is a smooth Fréchet manifold and it is a submanifold of $\mathcal{M}(L, M)$.*
2. *In addition to the assumptions of 1, suppose further that $P_1 = D_1^* D_1 : V_1 \rightarrow V_1$ is elliptic. Then the moduli space $\mathcal{M}_L(\Phi)$ is a finite dimensional smooth manifold around $[\iota]$ and its dimension is equal to $\dim \ker(D_1)$.*

Proof. Consider the case 1. First, we extend above spaces and operators to those of class $C^{k,a}$, where $k \geq 1$ is an integer and $0 < a < 1$. Set

$$\begin{aligned} V_1^{k,a} &= C^{k,a}(L, \mathcal{N}_\iota), & V_2^{k,a} &= C^{k,a}(L, \oplus_{i=1}^m \wedge^{k_i} T^*L), \\ U^{k,a} &= \{v \in V_1; v_x \in \mathcal{U} \text{ for any } x \in L\}, \\ F^{k,a} : U^{k,a} &\rightarrow V_2^{k-1,a}, & D_1^{k,a} &= (dF^{k,a})_0, \\ \mathcal{M}_L^{k,a}(\Phi) &= \{\iota \in C_{emb}^{k,a}(L, M); \iota^* \Phi = 0\} / \text{Diff}^{k,a}(L). \end{aligned}$$

Similarly, a neighborhood of $[\iota]$ in $\mathcal{M}_L^{k,a}(\Phi)$ is identified with that of 0 in $(F^{k,a})^{-1}(0)$ (in the C^1 sense).

We prove that $\mathcal{M}_L^{k,a}(\Phi)$ is smooth around $[\iota]$ in the sense of Banach. To apply the implicit function theorem, we prove the following.

Lemma 7.3. (a) $\text{Im}(D_1^{k,a}) \subset V_2^{k-1,a}$ is a closed subspace.

(b) $\text{Im}(F^{k,a}) \subset \text{Im}(D_1^{k,a})$.

(c) $D_1^{k,a} : V_1^{k,a} \rightarrow \text{Im}(D_1^{k,a})$ has a right inverse.

Proof. By the Hodge decomposition, we have

$$V_2^{k-1,a} = \ker P_2 \oplus D_1^{k,a}(V_1^{k,a}) \oplus (D_2^*)^{k,a}(V_3^{k,a}), \quad (7.1)$$

where $(D_2^*)^{k,a} : V_3^{k,a} \rightarrow V_2^{k-1,a}$ is a canonical extension of D_2^* . This is a L_2 -orthogonal decomposition and $\text{Im}(D_1^{k,a})$ is the orthogonal complement of $\ker P_2 \oplus (D_2^*)^{k,a}(V_3^{k,a})$. Thus we see (a).

We prove (b). For any $f \in U^{k,a}$, there exists a sequence $\{f_n\} \subset U$ such that $f_n \rightarrow f$. By $F(f_n) \in \text{Im}(F) \subset \text{Im}(D_1) \subset \text{Im}(D_1^{k,a})$, $F^{k,a}(f) = \lim_{n \rightarrow \infty} F(f_n)$ and (a), we see that $F^{k,a}(f) \in \text{Im}(D_1^{k,a})$.

We prove (c). Let G be the Green's operator of P_2 . Then for any $f \in \text{Im}(D_1^{k,a})$, we have $f = P_2 G(f) = D_1^{k,a}(D_1^*)^{k+1,a} G(f) + (D_2^*)^{k,a} D_2^{k+1,a} G(f)$. By (7.1), we deduce that

$$D_1^{k,a}(D_1^*)^{k+1,a} G(f) = f, \quad (D_2^*)^{k,a} D_2^{k+1,a} G(f) = 0. \quad (7.2)$$

Thus we see that $(D_1^*)^{k+1,a} G|_{\text{Im}(D_1^{k,a})} : \text{Im}(D_1^{k,a}) \rightarrow V_1^{k,a}$ is a right inverse of $D_1^{k,a} : V_1^{k,a} \rightarrow \text{Im}(D_1^{k,a})$. \square

By Lemma 7.3 (a), we obtain the smooth map $F^{k,a} : U^{k,a} \rightarrow \text{Im}(D_1^{k,a})$. The smoothness of this map is proved in [1, Theorem 2.2.15]. It is clear that $(dF^{k,a})_0 = D_1^{k,a} : V_1^{k,a} \rightarrow \text{Im}(D_1^{k,a})$ is surjective and $V_1^{k,a}$ is the direct sum of the kernel of $D_1^{k,a}$ and the image of the right inverse of $D_1^{k,a} : V_1^{k,a} \rightarrow \text{Im}(D_1^{k,a})$. By the proof of Lemma 7.3 (c), we have

$$V_1^{k,a} = X_1^{k,a} \oplus Y_1^{k,a},$$

where $X_1^{k,a} = \ker(D_1^{k,a})$ and $Y_1^{k,a} = (D_1^*)^{k+1,a} G \text{Im}(D_1^{k,a})$. Note that both spaces are closed in $V_1^{k,a}$.

Then we can apply the implicit function theorem. There exist an open neighborhood $A_1^{k,a} \subset X_1^{k,a}$ of 0, an open neighborhood $B_1^{k,a} \subset Y_1^{k,a}$ of 0, and a smooth mapping $\hat{G}^{k,a} : A_1^{k,a} \rightarrow B_1^{k,a}$ such that

$$(F^{k,a})^{-1}(0) \cap (A_1^{k,a} \oplus B_1^{k,a}) = \{x + \hat{G}^{k,a}(x); x \in A_1^{k,a}\},$$

which implies that $\mathcal{M}_L^{k,a}(\Phi)$ is smooth around $[\iota]$ in the sense of Banach.

Next, we prove that $\mathcal{M}_L(\Phi)$ is smooth around $[\iota]$ in the sense of Fréchet. The proof is an analogue of that of [11, Theorem 4.1]. The open set $A_1^{k,a}$ and the map $\hat{G}^{k,a}$ depend on k and a . We have to show that we can take $A_1^{k,a}$ and $\hat{G}^{k,a}$ “uniformly”. Namely, set

$$G^{k,a} = \hat{G}^{1,a}|_{A_1^{1,a} \cap V_1^{k,a}} : A_1^{1,a} \cap V_1^{k,a} \rightarrow B_1^{1,a}.$$

In the following, by shrinking $A_1^{1,a}$ if necessary, we prove that for any $k \geq 1$

- $\text{Im}(G^{k,a}) \subset Y_1^{k,a} = Y_1^{1,a} \cap V_1^{k,a}$,
- and $G^{k,a} : A_1^{1,a} \cap V_1^{k,a} \rightarrow Y_1^{k,a}$ is smooth in the sense of Banach.

Then we see that $\text{Im}(\hat{G}^{1,a}|_{A_1^{1,a} \cap V_1}) \subset Y_1^{1,a} \cap V_1$ and $\hat{G}^{1,a}|_{A_1^{1,a} \cap V_1}$ is smooth in the sense of Fréchet. Hence we see that $\mathcal{M}_L(\Phi)$ is smooth around $[\iota]$.

First, we show that $\text{Im}(G^{k,a}) \subset Y_1^{k,a}$ by the elliptic regularity theorem. For any $\gamma \in V_1$, define the second order differential operator $F_\gamma : V_2 \rightarrow V_2$ by

$$F_\gamma(\beta) = F(\gamma + D_1^* \beta) + D_2^* D_2 \beta.$$

Denote by $F_\gamma^{1,a}$ the extension of F_γ on $V_2^{1,a}$.

Since the linearization of F_0 at 0, which is given by $(dF_0)_0 = D_1 D_1^* + D_2^* D_2 = P_2$, is elliptic and the ellipticity is an open condition, we see that there exist an open neighborhood $\mathcal{U}_0 \subset V_1^{1,a}$ of 0 and an open neighborhood $\mathcal{V}_0 \subset V_2^{2,a}$ of 0 such that $(dF_\gamma^{1,a})_\beta$ is elliptic for any $(\gamma, \beta) \in \mathcal{U}_0 \times \mathcal{V}_0$. Set

$$\mathcal{U}_1 = (G^{1,a})^{-1}((D_1^*)^{2,a}(\mathcal{V}_0 \cap G(\text{Im}(D_1^{1,a}))) \cap B_1^{1,a}) \cap \mathcal{U}_0,$$

which is an open subset of $A_1^{1,a}$ because

$$(D_1^*)^{2,a}|_{G(\text{Im}(D_1^{1,a}))} : G(\text{Im}(D_1^{1,a})) \rightarrow (D_1^*)^{2,a} G(\text{Im}(D_1^{1,a})) = Y_1^{1,a}$$

is an isomorphism.

Lemma 7.4. *For any $k \geq 1$, we have*

$$G^{1,a}(\mathcal{U}_1 \cap V_1^{k,a}) \subset Y_1^{k,a}.$$

Proof. Let $\alpha \in \mathcal{U}_1 \cap V_1^{k,a}$. Since F is the first order differential operator, the differential operator F_α is of class $C^{k-1,a}$. By the definition of \mathcal{U}_1 , there exists $\beta \in \mathcal{V}_0 \cap G(\text{Im}(D_1^{1,a}))$ satisfying $G^{1,a}(\alpha) = (D_1^*)^{2,a}(\beta)$. Then

$$F_\alpha^{1,a}(\beta) = F^{1,a}(\alpha + (D_1^*)^{2,a} \beta) + (D_2^*)^{1,a} D_2^{2,a} \beta = 0$$

by the definition of $G^{1,a}$ and (7.2). Since $(\alpha, \beta) \in \mathcal{U}_0 \times \mathcal{V}_0$, $(dF_\alpha^{1,a})_\beta$ is elliptic. Hence Schauder theory implies that β is of class $C^{k+1,a}$. Thus $G^{1,a}(\alpha) = (D_1^*)^{2,a}(\beta)$ is of class $C^{k,a}$. \square

Next, we show that $G^{k,a}$ is a smooth map. Since $(dF^{1,a})_0|_{Y_1^{1,a}} = D_1^{1,a}|_{Y_1^{1,a}} : Y_1^{1,a} \rightarrow Z_2^{0,a} = D_1^{1,a}(V_1^{1,a})$ is an isomorphism and being an isomorphism is an open condition, there is an open neighborhood $\mathcal{U}_2 \subset V_1^{1,a}$ of 0 such that $(dF^{1,a})_\gamma|_{Y_1^{1,a}} : Y_1^{1,a} \rightarrow Z_2^{0,a}$ is an isomorphism for any $\gamma \in \mathcal{U}_2$. Set $\mathcal{U}_3 = \mathcal{U}_2 \cap \mathcal{U}_0$.

Lemma 7.5. *For any $\gamma \in \mathcal{U}_3 \cap V_1^{k,a}$,*

$$(dF^{1,a})_\gamma|_{Y_1^{k,a}} : Y_1^{1,a} \cap V_1^{k,a} = Y_1^{k,a} \rightarrow D_1^{k,a}(V_1^{k,a}) = Z_2^{0,a} \cap V_2^{k-1,a}$$

is an isomorphism.

Proof. The injectivity of $(dF^{1,a})_\gamma|_{Y_1^{k,a}}$ follows from the fact that $(dF^{1,a})_\gamma|_{Y_1^{k,a}}$ is a restriction of the isomorphism $(dF^{1,a})_\gamma|_{Y_1^{1,a}} : Y_1^{1,a} \rightarrow Z_2^{0,a}$. The equation $(dF^{1,a})_\gamma|_{V_1^{k,a}} = (dF^{k,a})_\gamma$ and the smoothness of $F^{k,a} : V_1^{k,a} \rightarrow D_1^{k,a}(V_1^{k,a})$ imply that $(dF^{1,a})_\gamma|_{Y_1^{k,a}}$ is continuous.

We prove that $(dF^{1,a})_\gamma|_{Y_1^{k,a}}$ is surjective. Take any $\mu \in D_1^{k,a}(V_1^{k,a})$. Since $(dF^{1,a})_\gamma|_{Y_1^{1,a}} : Y_1^{1,a} \rightarrow Z_2^{0,a}$ is an isomorphism, there exists $\beta \in G(\text{Im}(D_1^{1,a})) \subset V_2^{2,a}$ satisfying $(dF^{1,a})_\gamma((D_1^*)^{2,a}\beta) = \mu$. Now we have

$$\begin{aligned} (dF^{1,a})_\gamma((D_1^*)^{2,a}\beta) &= \frac{d}{dt} F^{1,a}(\gamma + t(D_1^*)^{2,a}\beta) \Big|_{t=0} \\ &= \frac{d}{dt} \left(F^{1,a}(\gamma + t(D_1^*)^{2,a}\beta) + (D_2^*)^{1,a} D_2^{2,a}(t\beta) \right) \Big|_{t=0} \\ &= (dF_\gamma^{1,a})_0(\beta) \end{aligned}$$

by (7.2). Since $\gamma \in \mathcal{U}_0 \cap V_1^{k,a}$, the differential operator $(dF_\gamma^{1,a})_0$ is the elliptic operator of class $C^{k-1,a}$. Hence by Schauder theory, β is of class $C^{k+1,a}$, which implies that $\mu = (dF^{1,a})_\gamma((D_1^*)^{2,a}\beta) \in (dF^{1,a})_\gamma(Y_1^{k,a})$. \square

Define the map $\tilde{G}^{1,a} : A_1^{1,a} \rightarrow V_1^{1,a} = X_1^{1,a} \oplus Y_1^{1,a}$ by $\tilde{G}^{1,a}(\alpha) = \alpha + G^{1,a}(\alpha)$. Set

$$\mathcal{U}_4 = (\tilde{G}^{1,a})^{-1}(\mathcal{U}_3) \cap \mathcal{U}_1,$$

which is an open set of $A_1^{1,a}$.

Lemma 7.6. *For any $k \geq 1$, $G^{k,a}|_{\mathcal{U}_4 \cap V_1^{k,a}} : \mathcal{U}_4 \cap V_1^{k,a} \rightarrow Y_1^{k,a}$ is smooth.*

Proof. We only have to prove that $G^{k,a}$ is smooth around any $\alpha_0 \in \mathcal{U}_4 \cap V_1^{k,a}$. Set $\gamma_0 = \tilde{G}^{1,a}(\alpha_0) = \alpha_0 + G^{1,a}(\alpha_0)$. By Lemma 7.4, $\gamma_0 \in \mathcal{U}_3 \cap V_1^{k,a}$. By Lemma 7.5, $(dF^{1,a})_{\gamma_0}|_{Y_1^{k,a}} : Y_1^{k,a} \rightarrow D_1^{k,a}(V_1^{k,a})$ is an isomorphism. Set $\tilde{X}_1^{k,a} = \ker(dF^{1,a})_{\gamma_0} \cap V_1^{k,a}$. Then we have

$$F^{1,a}(\gamma_0) = 0, \quad V_1^{k,a} = \tilde{X}_1^{k,a} \oplus Y_1^{k,a}.$$

Let $\tilde{\pi} : V_1^{k,a} = \tilde{X}_1^{k,a} \oplus Y_1^{k,a} \rightarrow \tilde{X}_1^{k,a}$ be the canonical projection and set $\tilde{\alpha}_0 = \tilde{\pi}(\alpha_0)$. This is a smooth mapping between Banach spaces. Applying the implicit function theorem to $F^{k,a} = F^{1,a}|_{V_1^{k,a}} : V_1^{k,a} \rightarrow D_1^{k,a}(V_1^{k,a})$, there exist an open

neighborhood $\tilde{U}_1^{k,a} \subset \tilde{X}_1^{k,a}$ of $\tilde{\alpha}_0$, an open set $\tilde{V}_1^{k,a} \subset Y_1^{k,a}$ and a smooth map $H^{k,a} : \tilde{U}_1^{k,a} \rightarrow \tilde{V}_1^{k,a}$ such that

$$(F^{k,a})^{-1}(0) \cap (\tilde{U}_1^{k,a} \oplus \tilde{V}_1^{k,a}) = \{\tilde{\alpha} + H^{k,a}(\tilde{\alpha}); \tilde{\alpha} \in \tilde{U}_1^{k,a}\}.$$

Now recall that for any $\alpha \in A_1^{1,a} \cap (\tilde{U}_1^{k,a} \oplus \tilde{V}_1^{k,a})$, we have $F^{1,a}(\alpha + G^{1,a}(\alpha)) = 0$ and $G^{k,a}(\alpha) = G^{1,a}(\alpha) \in Y_1^{k,a}$ by Lemma 7.4. Then there exists $\tilde{\alpha} \in \tilde{U}_1^{k,a}$ satisfying $\alpha + G^{k,a}(\alpha) = \tilde{\alpha} + H^{k,a}(\tilde{\alpha})$. Taking $\tilde{\pi}$ of both sides, we obtain $\tilde{\pi}(\alpha) = \tilde{\alpha}$, which implies that

$$G^{k,a}(\alpha) = \tilde{\pi}(\alpha) + H^{k,a}(\tilde{\pi}(\alpha)) - \alpha.$$

Thus $G^{k,a}|_{\tilde{U}_1^{k,a} \oplus \tilde{V}_1^{k,a}}$ is smooth. \square

By Lemma 7.6, it follows that $\hat{G}^{1,a}|_{\mathcal{U}_4 \cap V_1}$ is smooth in the sense of Fréchet, which implies that $\mathcal{M}_L(\Phi)$ is smooth around $[\iota]$.

Next, we prove that $\mathcal{M}_L(\Phi)$ is a submanifold of $\mathcal{M}(L, M)$ around $[\iota]$. Set $\mathfrak{U} = (\mathcal{U}_4 \cap V_1) \oplus Y_1$, where $Y_1 = Y_1^{1,a} \cap V_1$. Setting $X_1 = X_1^{1,a} \cap V_1$, we have $V_1 = X_1 \oplus Y_1$. Let $p : V_1 = X_1 \oplus Y_1 \rightarrow X_1$ be the canonical projection. Define the map $\psi : \mathfrak{U} \rightarrow \mathfrak{U}$ by $\psi(z) = z - \hat{G}^{1,a} \circ p(z)$. By Lemma 7.4, the image of ψ is contained in \mathfrak{U} . This is bijective and the inverse ψ^{-1} is given by $\psi^{-1}(z) = z + \hat{G}^{1,a} \circ p(z)$. Both mappings are smooth in the sense of Fréchet. It is clear that

$$\psi\left(\{\alpha + \hat{G}^{1,a}(\alpha); \alpha \in \mathcal{U}_4 \cap X_1\}\right) = \mathcal{U}_4 \cap X_1$$

since $\psi(\alpha + \hat{G}^{1,a}(\alpha)) = \alpha + \hat{G}^{1,a}(\alpha) - \hat{G}^{1,a}(\alpha) = \alpha$. Thus $\mathcal{M}_L(\Phi)$ is locally identified with the closed subspace X_1 of V_1 , which implies that the $\mathcal{M}_L(\Phi)$ is a submanifold of $\mathcal{M}(L, M)$ around $[\iota]$.

Finally, we prove the case 2. Since $P_1 = D_1^* D_1$ is elliptic, we have $\dim \ker(D_1) \leq \dim \ker P_1 < \infty$. Then we see the statement from the case 1. \square

Since the affine Legendrian condition is an open condition, we see as in the proof of [11, Theorem 3.4] that the moduli space of affine Legendrian submanifolds, namely, $\{[\iota] \in \mathcal{M}(L, M); \iota \text{ is affine Legendrian}\}$, is a smooth Fréchet manifold and it is open in $\mathcal{M}(L, M)$. Applying Proposition 7.2, we prove Theorem 1.6.

Proof of Theorem 1.6. Use the notation after Definition 7.1. The moduli space of special affine Legendrian embeddings of L is given by $\mathcal{M}_L(\text{Im}\psi)$. Fix any $[\iota] \in \mathcal{M}_L(\text{Im}\psi)$. Set $\mathcal{N}_\iota = \phi_{\iota*} TL \oplus \mathbb{R}\xi \circ \iota$. Define the map $F : U \rightarrow C^\infty(L)$ by

$$F(v) = *(\exp_v^*(\text{Im}\psi)),$$

where $*$ is the Hodge star operator of ι^*g . Then the linearization $(dF)_0$ of F at 0 is given by

$$(dF)_0(v) = *\iota^* L_v \text{Im}\psi = *(\iota^*(i(v)d\text{Im}\psi + di(v)\text{Im}\psi)).$$

By [9, Proposition 3.2], we have $d\psi = -(n+1)i\eta \wedge \psi$. Since ι is special affine Legendrian, we have $\iota^* \text{Re}(\psi) = \text{vol}_\phi[\iota]$. Then we compute

$$\iota^*(i(v)d\text{Im}\psi) = (n+1)(-\eta(v)\text{vol}_\phi[\iota] + \iota^*(\eta \wedge i(v)\text{Re}\psi)).$$

Denoting $v = \phi\iota_*Y + f\xi$ where $Y \in \mathfrak{X}(L)$, $f \in C^\infty(L)$, we have

$$\begin{aligned} i(v)\psi &= i(\phi\iota_*Y)i\left(r\frac{\partial}{\partial r}\right)\Omega|_{r=1} \\ &= i(J\iota_*Y)i\left(r\frac{\partial}{\partial r}\right)\Omega|_{r=1} \\ &= i \cdot i(\iota_*Y)i\left(r\frac{\partial}{\partial r}\right)\Omega|_{r=1} = i \cdot i(\iota_*Y)\psi. \end{aligned}$$

which implies that

$$\iota^*(i(v)\text{Re}\psi) = 0, \quad \iota^*d(i(v)\text{Im}\psi) = d(i(Y)\text{vol}_\phi[\iota]).$$

Then we obtain

$$\begin{aligned} D_1(v) &= (dF)_0(v) = *(-(n+1) * (\rho_\phi[\iota]f) + d * (\iota^*g(\rho_\phi[\iota]Y, \cdot))) \\ &= -(n+1)\rho_\phi[\iota]f - d^*(\iota^*g(\rho_\phi[\iota]Y, \cdot)). \end{aligned}$$

Via the identification

$$\begin{array}{ccc} C^\infty(L, \mathcal{N}_\iota) = & C^\infty(L, \phi\iota_*TL \oplus \mathbb{R}\xi \circ \iota) & \longrightarrow C^\infty(L) \oplus \Omega^1(L) \\ & \downarrow \Psi & \downarrow \Psi \\ & \phi\iota_*Y + f\xi & \longmapsto (\rho_\phi[\iota]f, \iota^*g(\rho_\phi[\iota]Y, \cdot)), \end{array}$$

the map D_1 is given by

$$C^\infty(L) \oplus \Omega^1(L) \ni (g, \alpha) \mapsto -(n+1)g - d^*\alpha \in C^\infty(L).$$

Since $D_1^*(h) = (-(n+1)h, -dh)$, we see that

$$D_1D_1^*(h) = (n+1)^2h + d^*dh,$$

which is clearly elliptic. We easily see that $\text{Im}(F) \subset \text{Im}(D_1)$ since $\text{Im}(D_1) = C^\infty(L)$. Then setting $D_2 = 0$ and $V_3 = \{0\}$, we can apply Proposition 7.2 to see that the moduli space of special affine Legendrian embeddings of L is an infinite dimensional smooth Fréchet manifold modeled on the Fréchet vector space $\{(g, \alpha) \in C^\infty(L) \oplus \Omega^1(L); (n+1)g + d^*\alpha = 0\} \cong \Omega^1(L)$. Note that we have $\Omega^1(L) = \{\alpha \in \Omega^1(L); d^*\alpha = 0\} \oplus dC^\infty(L)$ by the Hodge decomposition and $dC^\infty(L) = C^\infty(L)/\mathbb{R}$ is identified with the space of functions with integral 0.

Since the moduli space of affine Legendrian submanifolds is open in $\mathcal{M}(L, M)$ and special affine Legendrian submanifolds are affine Legendrian, the proof is done. \square

Remark 7.7. Applying Proposition 7.2 to the affine Lagrangian case, we can also deduce [11, Theorem 1.1].

References

- [1] P. D. Baier, Special Lagrangian Geometry, PhD thesis, University of Oxford, (2001).

- [2] V. Borrelli, Maslov form and J -volume of totally real immersions, *J. Geom. Phys.* 25 (1998), 271-290.
- [3] C. P. Boyer and K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, (2008).
- [4] B.-Y. Chen, *Geometry of submanifolds and its applications*, Science University of Tokyo, (1980).
- [5] R. Hamilton, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.* 7 (1982), 65-222.
- [6] R. Harvey and H. B. Lawson, Calibrated geometries, *Acta Math.* 148 (1982), 47-157.
- [7] T. Kajigaya, Second variation formula and the stability of Legendrian minimal submanifolds in Sasakian manifolds, *Tohoku Math. J. (2)* 65 (2013), 523-543.
- [8] J. D. Lotay and T. Pacini, Coupled flows, convexity and calibrations: Lagrangian and totally real geometry, [arXiv:1404.4227](https://arxiv.org/abs/1404.4227).
- [9] T. Moriyama, Deformations of special Legendrian submanifolds in Sasaki-Einstein manifolds, [arXiv:1306.2764](https://arxiv.org/abs/1306.2764).
- [10] H. Ono, Second variation and Legendrian stabilities of minimal Legendrian submanifolds in Sasakian manifolds, *Differential Geom. Appl.* 22 (2005), 327-340.
- [11] B. Opozda, A moduli space of minimal affine Lagrangian submanifolds, *Ann. Global Anal. Geom.* 41 (2012), 535-547.